Properties of Dominator of an M-Semigroup

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Abstract: In this paper we discuss few properties of a collection of a special type of element of an M-semigroup, namely dominator. In an M-semigroup a dominator may be empty or properly contained in it or equal to the semigroup itself.

Keywords: M-semigroup, dominator, idempotent elements, Rectangular Band.

I. INTRODUCTION
In this paper we find a position of a dominator D in an M-semigroup M. We also find a necessary and sufficient condition for the existence of the dominator, a necessary and sufficient condition for a dominator D = M. Further, we decompose the dominator D, in which case decomposed part is a semi-inflation of the dominator. We also discuss some properties of dominator of an M-semigroup.

II. PRELIMINARIES
1) A subset S’ of a semigroup S is said to be a sub semi group of S if S’ is a semigroup with the same binary operation of S.
2) A semigroup S is said to be right(left) singular if for all x, y in S, xy = y (xy = x) such a semigroup is also called a right(left) zero semigroup
3) Let X and Y be any two nonempty sets. Then the system (S = X x Y; * ) where (x, y) * (x’, y’) = (x, y’); For all x, x’ in X and y, y’ in y is a band. It is called a rectangular band on X x Y (3)
4) A decomposition of a semigroup S is meant a partition of S into union of disjoint subsemigroups S_i, where i ∈ A, an index set.

A decomposition as above is sometimes denoted by ∪_{i ∈ A} S_i ; it is also said that S is decomposed over A

5) Let S = ∪_{i ∈ A} S_i be a decomposition of a semigroup S into subsemigroups S_i over an index A. If for each (i, j) in AxA there exists an element k of A such that S_i S_j ⊆ S_k then A becomes thereby a band. It is then said that S is the union of the band A of semigroups S_i (i ∈ A); sometimes it is also said that “ S is a band A of semigroups S_i, i ∈ A’

6) If a semigroup S is the union of a band A of semigroups S_i (i ∈ A) then A is the homomorphic image of S under the homomorphism,

f : S → A, xf = i for x in S_i (i ∈ A) and the semigroups S_i (i ∈ A) are the congruence classes of S induced by the homomorphism f.

7) If f : S → A is a homomorphism of a semigroup S onto a band A then S is the union of the band A of semigroups

S_i = (i) f^{-1}, i ∈ A

8) If A is band of type ζ, S is a band A of semigroups S_i (i ∈ A), and each semigroup S_i (i ∈ A) is a semigroup of type ζ, then S is called as a ζ-band A of ζ-semigroup.

The concept that is being defined now is due to Clifford and Preston (1). Let B be a semigroup. With each i of B, associate a set G_i consisting i (i in B) which are mutually disjoint. Let G = ∪_{i ∈ B} G_i (i in B) and let the product in B be extended to a product in G by defining xy = ij if x is in Gi and y in Gj (i, j in B). Then G is a semigroup which is called an Inflation of B. The following result is also due to the above authors:

9) The definition of inflation as given below is due to Tamura (8).

Let B be a given semigroup. Let S be any semigroup. Then S is an inflation of B if and only if, S contains B as a semigroup, B contains a homomorphic image of S.
10) An element $d$ of a semigroup $S$ is called a dominator element of $S$ if $dyd = d$ for all $y$ in $S$. By $D$, the dominator of $S$, is meant the set of all dominator elements of $S$.

III. DOMINATOR OF AN M-SEMIGROUP

A. Definition
An element $x$ of a semigroup $S$ is said to be a dominator of $S$ if and only if $xyx = x$ for all $y \in S$ [2]. The set $D$ of all dominators of $S$ is called the dominator of $S$ denoted by $D$. The dominator of a semigroup may be empty.

1) Examples: The following are examples of M-semigroups in which the dominator $D = \phi$.

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The following are two examples of M-semigroups which contains a proper dominator $D$.

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Here $D = \{a, b\}$.

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Here $D = \{a, b\}$.

Examples of M-semigroups in which the dominator is itself is a right zero semigroup.

B. Lemma: A dominator $D \neq \emptyset$ of an M-semigroup $M \cong R \times S$ has the following properties:

(i) $D \cap R = \emptyset$ or $D = R = M$

(ii) $D \cap Me \cong D \cap Mf; \quad e, f \in R$

(iii) $D = \bigcup(D \cap Me)$. 

Proof: (i) If $D \cap R \neq \emptyset$, let $e$ belongs to $D \cap R$. For any $x$ belongs to $M$, $exe = xe$.

But, $exe = e$ since $e \in D$.

Therefore $xe = e$, for all $x$ belongs to $M$.

Therefore for any $a$ belongs to $M$, $ea = xea$.

That is, $a = xa$, for all $x$ belongs to $M$.

That is, every element of $M$ is a right zero element and hence $M$ is a right zero semigroup [2.2] which is a rectangular band.

Therefore $M = R = D$. i.e., $R \cap D = \emptyset$ or $R = M = D$.

(ii) Follows from for any ideal $I$ of an M-semigroup $M I \cap Me \cong I \cap Mf ; \quad e, f \in R$. [4]

A semigroup $S$ contains a dominator if and only if it contains an ideal $I$ which is a rectangular band. Then $I$ is the dominator of $S$ [2].

Each $I \cap Me, e \in R$ is a left ideal of $I$ [4].

C. Lemma

Every ideal $I$ of an M-semigroup $M \cong R \times S$ is a disjoint union of subsemigroup $I \cap Me, e \in R$. That is,

$I = \bigcup(I \cap Me).$ [4]

D. Lemma

A semigroup $S$ contains a dominator if and only if it contains an ideal $I$ which is a rectangular band. Then $I$ is the dominator of $S$ [1]. Follows from 3.3

The following lemmas gives the conditions for the existence of the dominator in an M-semigroup.

E. Lemma:

In an M-semigroup $M \cong R \times S$, if the dominator $D$ exists then $D \subseteq E \setminus R$ where $E$ is the set of idempotents of $M$.

Proof: Since $D$ is a rectangular band ideal,

$D \subseteq E$ being a rectangular band and $D \cap R = \emptyset$ being an ideal.

If $R = E$ and $D$ exists, then $D \cap E = \emptyset$ and $D \subseteq E$ implies $D = \emptyset$. 

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F. Lemma
In a left cancellative M-semigroup $M \cong R \times S$, $D = \emptyset$ or $D = M$.

1) Proof: If $M$ is a left cancellative then every idempotent of $M$ is a left identity. That is, $E = R$.

Therefore $D \subseteq E = R$.

From 3.2(i), $D = \emptyset$ or $D = M = R$.

G. Lemma
In an M-semigroup $M \cong R \times S$, if the dominator $D$ of $M$ is equal to $M$ then $M$ is left cancellative.

1) Proof: $D = M$ implies $D \cap R \neq \emptyset$ and $D \neq \emptyset$ implies $D = R$ by 4.8(i)

implies $D = M = R$.

That is $xyx = x = x^2$ for all $x, y$ belongs to $R = M$.

That is, if $xy = xz$, then $y = z$ since $x$ belongs to $R$.

Hence $M$ is left cancellative, and hence the lemma. From 3.6 and 3.7 we have:

H. Theorem
If an M-semigroup $M \cong R \times S$ has a nonempty dominator $D$, then $D = M$ if and only if $M$ is left cancellative.

I. Lemma
In an M-semigroup $M \cong R \times S$ if any one of the left identities $e$ of $R$ is primitive then $D \neq M$ implies $D = \emptyset$.

1) Proof: Let a particular $e \in R$ be primitive. For any idempotent $g$ of $Me$, $ge = g$ and $eg = g$.

Therefore, $ge = eg = g$.

That is, $e = g$, since $e$ is primitive.

Hence, $e$ is the only idempotent in $Me$.

Let $D \neq M$, if $D \neq \emptyset$, $D$ being the kernel of $M$, $D$ intersects all $Me$, $e \in R$ and $D \cap R = \emptyset$. This implies, there are idempotent elements other than $e$ in $Me$, for all $e$ belongs to $R$. This contradicts the property that $e$ is primitive.

Hence the lemma.

J. Theorem
An M-semigroup $M \cong R \times S$ contains a dominator $D$, if and only if $S$ contains a rectangular band ideal.

1) Proof: Let $S$ contains a rectangular band ideal $De$. That is $Me, e \in R$ contains a rectangular band ideal $De, e \in R$. Consider

$$D = \bigcup_{e \in R} De.$$ 

Since.

For any $x$ belongs to $De$, a $Me \cong Mf, De \cong Df \ (e, f \in R)$ and belongs to $Df$. 

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\[ ax = axx = (ax)x = (\text{element of } Me) \times e \in Df. \]
\[ xa = xaa \in Df. \]
\[ xax = xaxx = x(ax)x = x. \]

Therefore \( D \) is a rectangular band. For any \( xe \) belongs to \( De \), and \( af \) belongs to \( Mf, e, f \in R \),

\[ xe \cdot af = xaf = xf \cdot af \in Df, \quad \text{since } xf \in Df. \]
\[ af \cdot xe = axe \in De. \]

That is \( D \) is a rectangular band ideal and hence \( D \) is the dominator.

Conversely, let \( M \) contain a dominator \( D \). \( D \) is a rectangular band ideal of \( M \). That is,

\[ D = \bigcup (D \cap Me), \text{ for a fixed } e \in D \cap Me \subseteq Me. \]
\[ e \in R \]

Since \( D \) is an ideal of \( M \), \( D \cap Me \) is an ideal of \( Me \).

Let \( xe, ye \) belong to \( D \cap Me \).

Then \( xe \cdot xe = xe \), since \( xe \) belongs to \( D \).

\[ xe \cdot ye \cdot xe = xe, \quad \text{since } xe, ye \text{ belongs to } D. \]

Therefore \( D \cap Me \) is a rectangular band for all \( e \in R \).

Since \( S \cong Me \), \( S \) contains a rectangular band ideal.

Since the dominator of an \( M \)-semigroup is an ideal, we have the following:

**IV. CONCLUSION**

This paper discussed the “Properties of Domination of an \( M \) semi group”.

**REFERENCES**


