iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: ETPAM-2018 **Issue:** Conference **Month of publication:** March 2018

DOI:

Approximation Methods and Condition Sequences

J. R.V. Edward[1] C. Adlin Viola[2]

[1]*J.R.V. Edward, Department Mathematics, Scott Christian College, Nagercoil - 629 003, Tamil Nadu, India.*
[2]*Department of Mathematics, N.M.S. Kamaraj College of Education, Pazhavilai – 629 501, Kanyakumari District, Tamil Nadu, India.*

Abstract:The condition number of a bounded operator or a matrix A plays a vital role in the problem of solving the equation Ax = b. For an approximation method (A_n) for A, the behavior of the corresponding sequence ($k(A_n)$) of condition numbers – simply known as the condition sequence – has much impact on the behavior of A. In this paper an attempt is made to expose some of the connections between the applicability and stability of approximation methods and the corresponding condition sequences.

I. CONDITION NUMBERS

A. Definition : A bounded operator A defined on a Banach or Hilbert space X is said to be invertible if A is 1 – 1 and onto and A^{-1} is also a bounded operator.

For an invertible bounded operator A, the condition number $k(A)$ of A is defined as

$$k(A) = \|A\|\|A^{-1}\|. \quad (1)$$

Suppose A is an infinite matrix. Then A can be regarded as an operator in ℓ^2 , where ℓ^2 is the space of all scalar sequences $x = (x_1, x_2, x_3\ldots.)$ such that $\sum_{j=1}^{\infty}|x_j|^2$ is convergent. For $x = (x_1, x_2, x_3\ldots.) \in \ell^2$, the norm of x is defined as

$$\|x\| = \left(\sum_{j=1}^{\infty}|x_j|^2\right)^{1/2}. \quad (2)$$

The domain $D(A)$ of A is the set of all $x \in \ell^2$ such that $Ax \in \ell^2$.

A is said to be bounded if $D(A) = \ell^2$ and there exists $\alpha < \infty$ such that

$$A(x) \le \alpha\|x\| \;\; for\ all\ x \in \ell^2 . \quad (3)$$

We call a matrix A invertible if it is non singular; that is, if there exists a matrix A' such that

$$AA' = A'A = I, \quad (4)$$ where I is the identify matrix

and A^{-1} is bounded.

B. Definition : For a bounded matrix A, its norm is defined by

$$\|A\| = \sup\left\{\|Ax\|/x \in \ell^2\ \|x\| \le 1\right\}. \quad (5)$$

Now (1) serves as the definition of condition numbers for matrices as well.

1) *Note :*The norm given in (5) is the standard norm for matrices. However, there are other norms as well for matrices and condition numbers may be defined accordingly
2) *Remark :* We have $1 = \|I\ \| \le \|AA^{-1}\| \le \|A\|\ A'\| = k(A)$. Thus $k(A) \ge 1$ for any A.

C. Condition Sequences

Suppose (A_n) is a sequence of bounded invertible operators (matrices) such that $A_n \to A$ in some sense. Discussion of theconnection between the behavior of A and the corresponding condition sequence ($k(A_n)$) is a vital area of research.

One of the basic questions before us is that whether $k(An)$ converges to $k(A)$, when $k(A) < \infty$? The answer is yes if both (A_n) converges to A and A_n^{-1} converges to A^{-1} in norm.

In that case, $\|A_n\| \to \|A\|$ *and* $\|A_n^{-1}\| \to \|A^{-1}\|$, as 'norm' is continuous.

So, $k(A_n) = \|A_n\|\|A_n^{-1}\| \to \|A_n\|\|A^{-1}\| = k(\mathrm{A}).$

But, norm convergence cannot be expected always. Let us try to answer the question for approximation methods.

D. Definition

Let H be an infinite dimensional Hilbert space and A be a bounded operator on H. let (H_n) be an increasing sequence of finite dimensional subspaces of H and (A_n) be a sequence of bounded operators, with $A_n \in B(H_n)$, for each n, such that $A_nP_n \to A$ pointwise. Then, the sequence (A_n) is called an approximation method for A.

Here P_n is the projection operator on H with $R(P_n) = H_n$ for each n, where $R(P_n)$ denotes the range of P_n. An approximation method (A_n) is said to be meaningful if A_n is invertible in $B(H_n)$ for all $n \geq n_0$ for some positive integer n_o. A meaningful approximation method (A_n) is said to be a stable method if

$$\sup_{n\geq n_0} \|A_n^{-1}\| < \infty .$$

By passing to a subsequence of (A_n), if necessary, we may assume that (A_n) is invertible for all n for a stableorapplicable approximation method (A_n). (6)

An approximation method (A_n) is said to be an applicable method for A if $A_n^{-1}P_n y \to A^{-1}y$ for all $y \in H$.

The same definitions can be generalized to study approximation methods for densely defined closed (possibly unbounded) operators A as well. But, choosing an appropriate convergence is a challenging task here, since one cannot expect the usual convergence for unbounded operators. Now let us define the condition sequence corresponding to an approximation method.

E. Definition

Let (A_n) be a meaningful approximation method for A. The sequence $(k(A_n)) = (k(A_1), k(A_2), \ldots.)$ is called the condition sequence corresponding to (A_n).

If $(k(A_n))$ converges to $k(A)$, we may say that (A_n) is a well – conditioned approximation method for A.

1) *Theorem:* Let $A \in B(H)$. Put $A_n = AP_n|H_n$, where $H_1 \subset H_2 \subset \ldots \subset H$ and $\cup H_n$ is dense in H. Then, $\lim_{x\to\infty} \|A_n\| = \|A\|$.

2) *Proof* : As $\cup H_n$ is dense in H, $A_nP_nx \to Ax$ *for all* $x \in H$.

We claim that $\|A_n\| \to \|A\|$.

Let $\in > 0$ be given. Then, $\|A\| - \in < \|A\|$. Since $\|A_n\| = \sup\{\|Ax\| / x \in H, \|x\| \leq 1\}$, there exists $x_0 \in H$ with $\|x_0\| \leq 1$ such that

$$\|A_n\| - \in < \|Ax_0\|. \qquad (8)$$

But $Ax_0 = \lim_{x\to\infty} A_nP_n\, x_0$.

$$So, \|Ax_0\| = \left\|\lim_{n\to\infty} A_nP_n\, x_0\right\| = \lim_{x\to\infty} \|A_nP_n\, x_0\|. \qquad (9)$$

From (8) and (9), $\lim_{n\to\infty} \|A_nP_n\, x_0\| > \|A\| - \in .$

Hence, there exists a positive integer n_o such that

$$\|A_n P_n x_0\| > \|A\| - \in \; for\, all\, n \geq n_0. \qquad (10)$$

But $\|A_n P_n x_0\| \leq \|A_n P_n\| \|x_0\| \leq \|A_n P_n\| \leq \|A_n\| \|P_n\| \leq \|A_n\|$, since $\|P_n\| \leq 1 and \|x_0\| \leq 1$.

Therefore, (10) becomes

$$\|A\| - \in < \|A_n P_n\| \leq \|A_n\| \text{ for all } n \geq n_0 \qquad (11)$$

Now, Fix n.

For $x \in H_n$, $P_n x = x$.

So, $A_n x = P_n A P_n x = P_n A x$.

Therefore, $\|A_n x\| = \|P_n A x\| \leq \|P_n\| \|Ax\| \leq \|Ax\|, as \|P_n\| \leq 1$.

That is , $\|A_n\| \leq \|A\|$. Thus, $\|A_n\| \leq \|A\| \; for\, all\, n$. (12)

From (11) and (12), $\|A_n\| - \in \leq \|A_n\| \leq \|A\| < \|A\| + \in$ for all $n \geq n_0$.

Hence $\lim_{n\to\infty} \|A_n\| = \|A\|.$ □

The sequence (A_n), where $A_n = AP_n|H_n$ is known as the projection method.

Let $A \in B(H)$ be invertible. Put $A_n = AP_n|H_n$ where $H_1 \subset H_2 \subset \ldots\ldots \subset H$ and $\cup H_n$ is dense in H. Suppose (A_n) is an applicable method for A. Then, (A_n) is well conditioned.

By Theorem 2.3, $\lim_{n\to\infty} \|A_n\| = \|A\|.$

As (A_n) is an applicable method for $A, A_n^{-1} P_n x \to A^{-1} x$ for all $x \in H$.

Then, replacing A_n by A_n^{-1} and $A \; by\, A^{-1}$ in the proof of Theorem 2.3 we get

$$\lim_{n\to\infty} \|A_n^{-1}\| = \|A^{-1}\|.$$

Now, $\lim_{n\to\infty} \|A_n\| \|A_n^{-1}\| = \left(\lim_{x\to\infty} \|A_n\|\right)\left(\lim_{x\to\infty} \|A_n^{-1}\|\right) = \|A\| \|A^{-1}\|.$

That is, $\lim_{n\to\infty} k(A_n) = k(A)$ and hence (A_n) is a well conditioned approximation method. □

Suppose (A_n) is an approximation method for A. that is, $A_n P_n \to A$ strongly. Then, (A_n) is an applicable method for A if and This is a basic result for approximation methods. One can refer [2] for a proof.

Using this result we are able to prove the following result for condition sequences of monotonic apporixation methods

E. Definition

An approximation method (A_n) is said to be monotonic if $\|A_n\| \leq \|A_m\|$ for n < m.

F. Theorem

Let (A_n) be a monotonic approximation method for A. Then, (A_n) is a stable method if and only if $(k(A_n))$ is a bounded sequence.

1) Proof: Suppose (A_n) is stable.

Without loss of generality, we may assume that A_n is invertible for all n and $\|A_n^{-1}\| \leq \alpha$.

Since (A_n) is an approximation method for A, $A_n P_n x \to Ax$ *for all* $x \in X$.

By the Uniform Boundedness Theorem, $\|A_n\| \le \beta$ for all n, for some $\beta < \infty$.

Hence $\|A_n\|\|A_n^{-1}\| \le \beta\alpha$ for all n and ($k(A_n)$) is bounded.

Conversely suppose that ($k(A_n)$) is bounded.

Then there exists $\gamma \subset \infty$ such that $k(A_n) \le \gamma$ for all n.

That is $\|A_n\|\|A_n^{-1}\| \le \gamma$ (13)

As (A_n) is monotonic, $\|A_1\| \le \|A_2\| \le ....$

Thus, $\|A_n\| \ge \|A_1\| > 0$ for all n, since $A_1 \ne 0$, *as* A_1 is invertible.

Now (13) implies, $\|A_n^{-1}\| \le \frac{\gamma}{\|A_n\|} \le \frac{\gamma}{\|A_1\|}$ for all n.

Thus (A_n) is stable. □

Hence (A_n) is an applicable method for A.

REFERENCES

[1] W. Arveson - C* - algebras and Numerical Linear Algebra. J. Funct. Anal. Vol. 122, No. 2,1994, pp. 333-360.

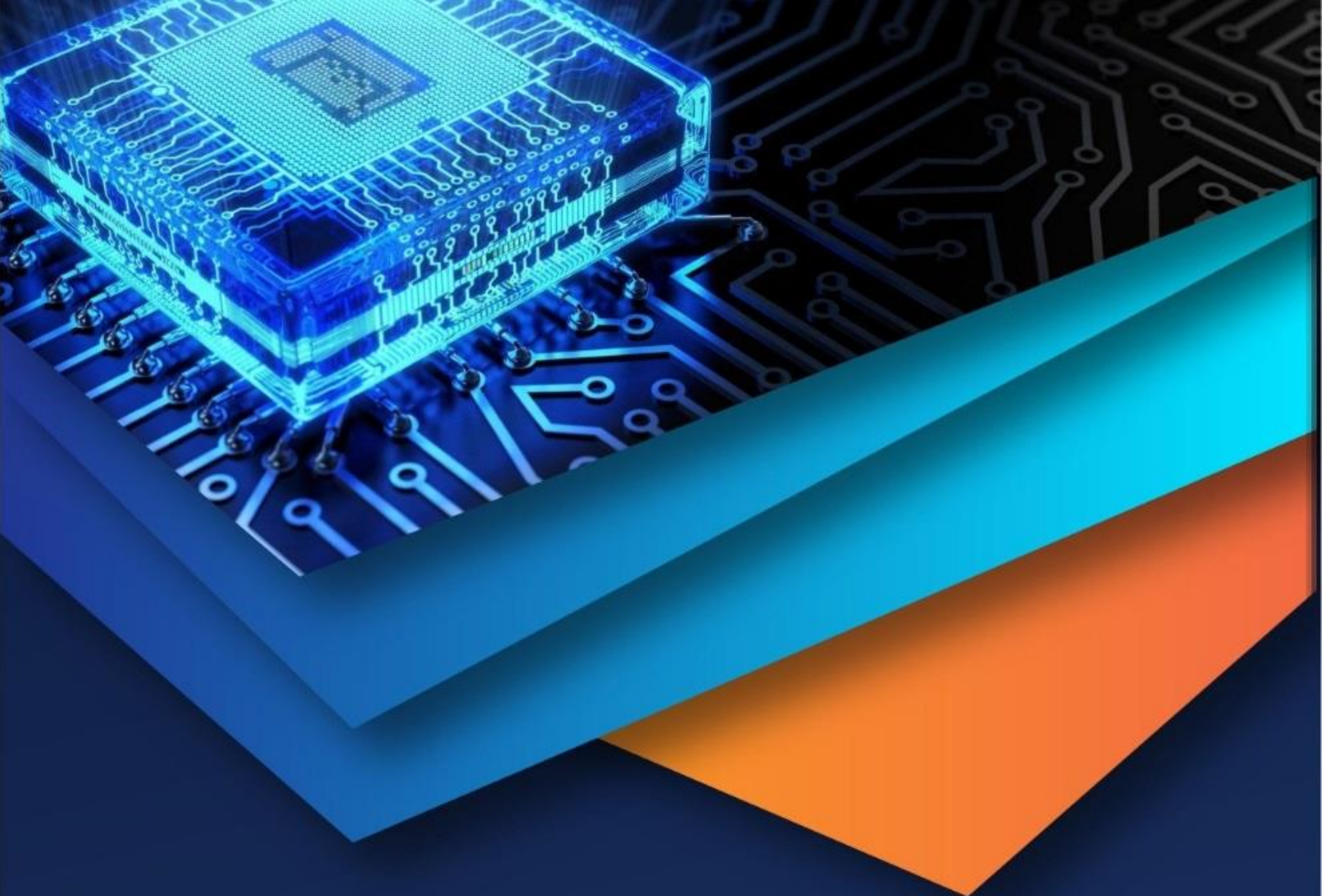
[2] A. Bottcher and S.M. Grudsky:– Toeplitz Matrices, Asymptotic Linear Algebra and Functional Analysis. Hindustan Book Agency, New Delhi, 2000.

[3] J.R.V. Edward: On the Finite Section Method for Unbounded self – Adjoint operators, J.Analysis, Vol. 142 (2006), P. 69 – 78.

[4] E. Kreyszig : Introductory functional Analysis with Applications, John wiley & sons, New York 1978.

[5] B.V. Limaye: Functional Analysis, Wiley Eastern Ltd., New Delhi, 1981.

[6] V. Sundarapandian : Numerical Linear algebra, Prentice Hall of India, Delhi 2014.



doi crossref
10.22214/IJRASET

INDEX COPERNICUS
45.98

ISRA JIF
IMPACT FACTOR:
7.129

TOGETHER WE REACH THE GOAL
IMPACT FACTOR:
7.429

ISI

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY