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# ε-Best Approximation and E- Orthogonality

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***Abstract: The purpose of this paper is to study the concept of ε-Best approximation and ε-orthogonality. I discussed their properties and noted that are similar to the properties of best approximation.***
***Keywords: ε-Best approximation, normed linear spaces, proximinal, ε-orthogonality, convex.***

## I. INTRODUCTION

The theory of best approximation is an important topic in functional analysis. It is a very extensive field which has various applications

What do we mean by "***Best approximation"*** in normed linear spaces?

To explain this, let X be a normed linear space, and let G be a nonempty subset of X. An element $g_0 \in G$ is called a best approximation to x from G if $g_0$ is closest to x from among all the elements of G.

That is, $\| x-g_0 \| \leq \| x-g \|$ for all $g \in G$.

The set of all such elements $g_0 \in G$ are called a ***best approximation*** to $x \in X$ is denoted by $P_G(x)$.

If $P_G(x)$ contains at least one element, then the subset G is called a ***proximinal*** set. If each element $x \in X$ has a unique best approximation in G, then G is called a ***Chebyshev*** set of X.

The theory of approximation is mainly concerned with the following fundamental questions.

1) *(Existence of best approximation)* Which subsets are proximinal?
2) *(Uniqueness of best approximation)* Which subsets are Chebyshev?
3) *(Characterization of best approximation)* How to recognize when a given $y \in G$ is a best approximation to x or not?
4) *(Error of approximation)* How to compute the error of approximation d(x, G)?
5) (*Computation of best approximation*) How to describe some useful algorithms for actually computing best approximation?
6) (*Continuity of best approximation*) How does the set of all best approximation vary as a function of x or (G)?

### A. Definition 1.1[1]

Let G be a nonempty subset of a real normed linear space E and let an element $f \in E$ be given. The problem of ***best approximation*** is to determine an element $g_f \in G$ such that

$$\| f- g_f \| = \inf_{g \in G} \|f - g\|$$

such an element is called a *best approximation* to f from G *,and*

$$d(f, G) = \inf_{g \in G} \|f - g\|$$ is called the *minimal deviation* off from G.

The set of all elements $g_0 \in G$ that are called best approximation to $x \in X$ is

$$P_G(x) = \{ g_0 \in G: \| x - g_0 \| \leq \| x - g \| \text{ for all } g \in G \}$$

Hence $P_G$ defines a mapping from X into the power set of G is called the ***metric projection*** onto G, (other names nearest point mapping, proximity map)

### B. Remark 1.2[1]

The set $P_G(x)$ of all *best approximation* to $x \in X$ can be written as

$$P_G(x) = \{g_0 \in G: \| x-g_0 \| = d(x, G)\}$$

*C. Definition 1.3.[3]*

A set S, in a linear space is ***convex*** .if $s_{1,}\ s_2 \epsilon S$ implies that

$$\lambda_1 s_1 + \lambda_2 s_2 \epsilon S$$

If $\lambda_1$ and $\lambda_2$ are non negative and $\lambda_1 + \lambda_2 = 1$

If S is empty or consists of one point, then it is clearly *convex*

*D. Definition 1.4[1]*

If $P_G(x)$ contains at least one element, then the subset G is called a *proximinal set.*

In other words, if $P_G(x) \neq \varphi$ then G is called a *proximinal set*

The term *proximinal set* (is a combination of proximity and maximal)

*E. Definition 1.5[1] (Quasi-Orthogonal Set)*

Let X be a normed linear space, and G a nonempty subset of X. Then we say that G is *quasi-orthogonal set* if $G \perp_B \hat{G}$, that is $g \perp_B \hat{G}$ for every $g \epsilon G$.

where $\hat{G} = \{x \epsilon X ;\| x \|= d(x, G)\} = \{x \epsilon X : x \perp_B G\}$.

*F. Remark 1.6[1]*

In a Hilbert space, any closed subspace is quasi-orthogonal.

*Proof:*

Let H be a Hilbert space and G a closed subspace of H.

Then $\hat{G} = G^{\perp} = \{y \epsilon H : < x, y > = 0, \text{ for all } x \epsilon G\}$. Then $G \perp \hat{G}$.

Therefore G is quasi-orthogonal subspace of H.

*G. Definition 1.7[2]*

Let X be a normed linear space and G be a subset of X, and $\varepsilon > 0$. A point $g_0 \epsilon G$ is said to be $\varepsilon$-best approximation for $x \epsilon X$ if and only if

$\| x - g_0 \| \leq \| x - g \| + \varepsilon$ for all $g \epsilon G$

*H. Remark 1.8[2]*

For $x \epsilon X$, the set of all $\varepsilon$-Best approximation of x in G is denoted by

$P_G(x, \varepsilon)$, in other words,

$P_G(x, \varepsilon) = \{g_0 \epsilon G: \| x - g_0 \| \leq \| x - g \| + \varepsilon \text{ for all } g \epsilon G\}$.

*I. Theorem 1.9[2]*

Let G be a subspace of a normed linear space X. Then $P_G(x, \varepsilon)$ is bounded.

***Proof*:**

Let $g_1, g_2 \epsilon P_G(x, \varepsilon)$, then $\| x - g_1 \| \leq \| x - g \| + \varepsilon$ for all $g \in G$, and

$$\| x - g_2 \| \leq \| x - g \| + \varepsilon \text{ for all } g \in G$$

$$\text{Now, } \| g_1 - g_2 \| = \| g_1 - x + x - g_2 \| \leq \| x - g_1 \| + \| x - g_2 \|$$

$$\leq \| x - g \| + \varepsilon + \| x - g \| + \varepsilon = 2 \| x - g \| + 2\varepsilon = k,$$

so we have $\| g_1 - g_2 \| \leq k$ where $k = 2d(x, G) + 2\varepsilon$.

Therefore, $P_G(x, \varepsilon)$ is bounded.

Hence the proof

*J. Theorem 1.10[2]*

Let G be a subspace of normed linear space X, and $x \in X$. Then $P_G(x, \varepsilon)$ is convex.

*Proof:*

Let $g_1, g_2 \in P_G(x, \varepsilon)$, and $0 \le \lambda \le 1$, then $\| x - g_1 \| \le \| x - g \| + \varepsilon$ for all $g \in G$, and

$\| x - g_2 \| \le \| x - g \| + \varepsilon$ for all $g \in G$

Now, $\| x - (\lambda g_1 + (1 - \lambda) g_2) \| = \| x - \lambda g_1 - g_2 + \lambda g_2 \|$

$$= \| x - \lambda g_1 - g_2 + \lambda g_2 + \lambda x - \lambda x \|$$
$$= \| \lambda(x - g_1) + (1 - \lambda)(x - g_2) \|$$
$$\le \| x - g_1 \| + (1 - \lambda) \| x - g_2 \|$$
$$\le \lambda(\| x - g \| + \varepsilon) + (1 - \lambda)(\| x - g \| + \varepsilon)$$
$$= \| x - g \| + \varepsilon.$$

Thus, $\lambda g_1 + (1 - \lambda) g_2 \in P_G(x, \varepsilon)$.

Hence $P_G(x, \varepsilon)$ is convex.

Hence the proof

*K. Definition 1.11.[2] ($\varepsilon$-orthogonality)*

Let X be a normed linear space, $\varepsilon > 0$, and $x, y \in X$. We call x is ***ε- orthogonal*** to y and is denoted by $x \perp_\varepsilon y$ if and only if

$\| x + \alpha y \| + \varepsilon \ge \| x \|$ for all scaler $\alpha$ with $| \alpha | \le 1$

For subsets $G_1$, $G_2$ of X, $G_1 \perp_\varepsilon G_2$ if and only if, $g_1 \perp_\varepsilon g_2$ for all $g_1 \in G_1$, $g_2 \in G_2$.

*L. Theorem: 1.12[2]*

Let X be a normed linear space, G be a subspace of X, and $\varepsilon > 0$.Then for all $x \in X$,

$g_0 \in P_G(x, \varepsilon)$ if and only if $(x - g_0) \perp_\varepsilon G$.

*Proof:*

(=>) Suppose $g_0 \in P_G(x, \varepsilon)$. Put $g_1 = g_0 - \alpha g$ for $g \in G$ and $| \alpha | \le 1$.

Since $g_0 \in P_G(x, \varepsilon)$ and $g_1 \in G$ so, then, $\| x - g_0 \| \le \| x - g_1 \| + \varepsilon$, then

$\| x - g_0 \| \le \| x - (g_0 - \alpha g) \| + \varepsilon$, and this implies that

$\| x - g_0 \| \le \| (x - g_0) + \alpha g \| + \varepsilon$.

Therefore, $(x - g_0) \perp_\varepsilon G$.

(<=) Let $(x - g_0) \perp_\varepsilon G$, then for all $\alpha$ with $| \alpha | \le 1$ and $g_1 \in G$

we have,

$\| x - g_0 \| \le \| x - g_0 + \alpha g_1 \| + \varepsilon$

For any $g \in G$ by putting $g_1 = g_0 - g$ and $\alpha = 1$, the last inequality implies,

$\| x - g_0 \| \le \| x - g \| + \varepsilon$

Therefore, $g_0 \in P_G(x, \varepsilon)$

Hence the proof

*M. Notation 1.13*

Let X be a normed linear space, and G a subspace of X, and for $\varepsilon > 0$, let

$P_G^{-1}(0, \varepsilon) = \{x \in X : \| x \| \le \| x - g \| + \varepsilon \text{ for all } g \in G\} = \{x \in X : x \perp_\varepsilon G\}$

Then, $\hat{G}_\varepsilon = \{x \in X : x \perp_\varepsilon G\}$.

*N. Lemma 1.14[2]*

Let G be a subspace of a normed linear space X. Then for all $x \in X$ and all $\varepsilon > 0$,

we have, $g_0 \in P_G(x, \varepsilon)$ if and only if $(x - g_0) \in \hat{G}_\varepsilon$

**Proof:**

$g_0 \in P_G(x, \varepsilon)$ if and only if by [Theorem1.12], $(x - g_0) \perp_\varepsilon \hat{G}_\varepsilon$ if and only if $(x - g_0) \in \hat{G}_\varepsilon$.

*O. Corollary 1.15*

Let G be a subspace of a normed linear space X, and let $\varepsilon > 0$, $x \in X$. Then,

$P_G(x, \varepsilon) = G \cap (x - \hat{G}_\varepsilon)$

*Proof:*

$g_0 \in G \cap (x - \hat{G}_\varepsilon)$ if and only if $g_0 \in G$, and $g_0 \in (x - \hat{G}_\varepsilon)$ if and only if
$g_0 \in G$ and $g_0 = x - \hat{g}$, where $\hat{g} \in \hat{G}_\varepsilon$ if and only if $g_0 \in G$, $\hat{g} = (x - g_0) \in \hat{G}_\varepsilon$
if and only if $g_0 \in P_G(x, \varepsilon)$ by [ Lemma 1.14].
Therefore, $P_G(x, \varepsilon) = G \cap (x - \hat{G}_\varepsilon)$
Hence the proof

*P. Theorem 1.16*

Let G be a subspace of a normed linear space X, $\varepsilon > 0$, and $\varepsilon \geq \alpha$. Then,
$\hat{G} \subseteq \hat{G}_\alpha \subseteq \hat{G}_\varepsilon$, and therefore $\cap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$

*Proof*:

Let $x \in \hat{G}$, then $\| x \| \leq \| x - g \|$ for all $g \in G$.
Now $\| x \| \leq \| x - g \| \leq \| x - g \| + \alpha$ [$\alpha > 0$], so, we have $x \in \hat{G}_\alpha$.
Hence $\hat{G} \subseteq \hat{G}_\alpha$ ………(1)
Let $x \in \hat{G}_\alpha$, then $\| x \| \leq \| x - g \| + \alpha \leq \| x - g \| + \varepsilon$ [$\varepsilon > \alpha$], this implies that
$x \in \hat{G}_\varepsilon$ , and so, $\hat{G}_\alpha \subseteq \hat{G}_\varepsilon$ ……..(2)
(1) and (2) together imply that $\hat{G} \subseteq \hat{G}_\alpha \subseteq \hat{G}_\varepsilon$,
Now, we show $\cap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$
From above we have $\hat{G} \subseteq \cap_{\varepsilon > 0} \hat{G}_\varepsilon$
conversely, let $x \in \cap_{\varepsilon > 0} \hat{G}_\varepsilon$ ,
Then for all $\varepsilon > 0$, $0 \leq \| x \| \leq \| x - g \| + \varepsilon$ for all $g \in G$, then for all $n \in N$,
$0 \leq \| x \| \leq \| x - g \| + \frac{1}{n}$ for all $g \in G$:
As $n \to \infty$, $\| x \| \leq \| x - g \|$ for all $g \in G$, then $x \in \hat{G}$,
and so,
$\cap_{\varepsilon > 0} \hat{G}_\varepsilon \subseteq \hat{G}$
Therefore $\cap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$
Hence the proof.

*Q. Lemma 1.17*

Let G be a subspace of a normed linear space X. Then.

*1)* If $\varepsilon > 0$, $x, g \in X$ and $x \perp_\varepsilon g$, then $x \perp_\delta g$ for all $\delta \geq \varepsilon$.
*2)* If $x, g \in X$ and $x \perp_B g$, then $x \perp_\varepsilon g$ for all $\varepsilon > 0$.
*3)* If $x \in X$, and $\varepsilon > 0$, then $0 \perp_\varepsilon x$, $x \perp_\varepsilon 0$.
*4)* If $x \perp_\varepsilon g$ and $|\beta| < 1$, then $\beta x \perp_\varepsilon \beta g$.

*Proof:*

(a) Let $\varepsilon > 0$, $x, g \in X$ and $x \perp_\varepsilon g$, then by [Definition 1.11] we have
$\| x \| \leq \| x + \alpha g \| + \varepsilon$, where $| \alpha | \leq 1$ and $\varepsilon > 0$
Then, $\| x \| \leq \| x + \alpha g \| + \varepsilon \leq \| x + \alpha g \| + \delta$, [since $\delta \geq \varepsilon$]
Therefore, $x \perp_\delta g$
(b) Let $x, g \in X$ and $x \perp_B g$, then $\| x \| \leq \| x + \alpha g \|$ for all $\alpha \in \mathbb{R}$
Since $\varepsilon > 0$, then $\| x \| \leq \| x + \alpha g \| \leq \| x + \alpha g \| + \varepsilon$ for all $| \alpha | \leq 1$
Hence $x \perp_\varepsilon g$ for all $\varepsilon > 0$
(c) Let $x \in X$ and $\varepsilon > 0$, then $\| 0 \| \leq \| 0 + \alpha x \| + \varepsilon$, and so $0 \perp_\varepsilon x$.
We have also $\| x \| \leq \| x \| + \varepsilon$, then $\| x \| \leq \| x + \alpha 0 \| + \varepsilon$,
Hence $x \perp_\varepsilon 0$.
(d) Let $x \perp_\varepsilon g$, and $| \beta | < 1$, then $\| x \| \leq \| x + \alpha g \| + \varepsilon$.
Multiply both sides by $| \beta |$,
we get $|\beta| \| x \| \leq \| \beta x + \beta \alpha g \| + | \beta | \varepsilon$
$\leq \| \beta x + \alpha_1 g \| + \varepsilon$, and so
$\| \beta x \| \leq \| \beta x + \alpha_1 g \| + \varepsilon$
Therefore, $\beta x \perp_\varepsilon \beta g$
Hence the proof

*R. Theorem 1.18*

Let G be a subspace of a normed linear space X. If $x \in X$, $\varepsilon > 0$ and $\delta \geq \varepsilon$, then $P_G(x, \varepsilon) \subsetneq P_G(x, \delta)$.

*Proof:*

Let $g_0 \in P_G(x, \varepsilon)$. Then by [Definition 1.7], we have

$\| x - g_0 \| \leq \| x - g \| + \varepsilon$ for all $g \in G$ and $\varepsilon > 0$

Then $\| x - g_0 \| \leq \| x - g \| + \varepsilon \leq \| x - g \| + \delta$

[since $\delta > \varepsilon$], then, $g_0 \in P_G(x, \delta)$.

Therefore $P_G(x, \varepsilon) \subsetneq P_G(x, \delta)$

Hence the proof

## II. CONCLUSION

Here, I conclude my paper as ε-Best approximation and ε- orthogonality has the properties which are similar to the properties of best approximation.

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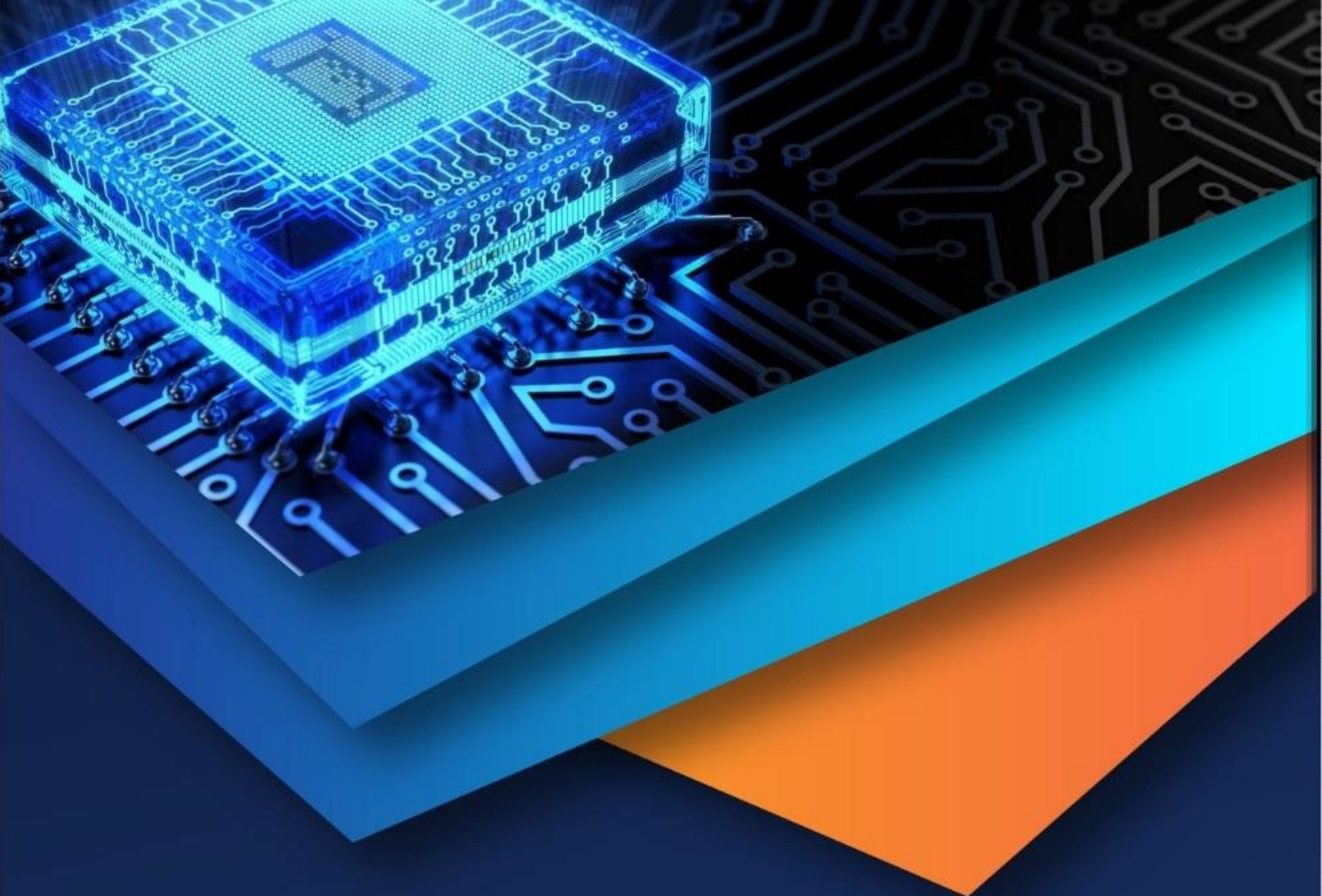

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