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# Emerging Trends in Pure and Applied Mathematics(ETPAM-2018)- March 2018 <br> $\varepsilon$-Best Approximation and E- Orthogonality 

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#### Abstract

The purpose of this paper is to study the concept of e-Best approximation and $\varepsilon$-orthogonality. I discussed their properties and noted that are similar to the properties of best approximation.


Keywords: $\varepsilon$-Best approximation, normed linear spaces, proximinal, $\varepsilon$-orthogonality, convex.

## I. INTRODUCTION

The theory of best approximation is an important topic in functional analysis. It is a very extensive field which has various applications
What do we mean by "Best approximation" in normed linear spaces?
To explain this, let $X$ be a normed linear space, and let $G$ be a nonempty subset of $X$. An element $g_{0} \in G$ is called a best approximation to x from G if $\mathrm{g}_{0}$ is closest to x from among all the elements of G .
That is, $\left\|x-g_{0}\right\| \leq\|x-g\|$ for all $g \in G$.
The set of all such elements $\mathrm{g}_{0} \in \mathrm{G}$ are called a best approximation to $\mathrm{x} \in \mathrm{X}$ is denoted by $\mathrm{P}_{\mathrm{G}}(\mathrm{x})$.
If $\mathrm{P}_{\mathrm{G}}(\mathrm{x})$ contains at least one element, then the subset G is called a proximinal set. If each element $\mathrm{x} \in \mathrm{X}$ has a unique best approximation in $G$, then $G$ is called a Chebyshev set of X .
The theory of approximation is mainly concerned with the following fundamental questions.

1) (Existence of best approximation) Which subsets are proximinal?
2) (Uniqueness of best approximation) Which subsets are Chebyshev?
3) (Characterization of best approximation) How to recognize when a given $\mathrm{y} \epsilon \mathrm{G}$ is a best approximation to x or not?
4) (Error of approximation) How to compute the error of approximation $\mathrm{d}(\mathrm{x}, \mathrm{G})$ ?
5) (Computation of best approximation) How to describe some useful algorithms for actually computing best approximation?
6) (Continuity of best approximation) How does the set of all best approximation vary as a function of $x$ or $(\mathrm{G})$ ?
A. Definition 1.1[1]

Let $G$ be a nonempty subset of a real normed linear space $E$ and let an element $f \in E$ be given. The problem of best approximation is to determine an element $g_{f} \in G$ such that

$$
\left\|f-g_{f}\right\|=\inf _{\mathrm{g} \in \mathrm{G}}\|f-\mathrm{g}\|
$$

such an element is called a best approximation to f from G , and

$$
\mathrm{d}(\mathrm{f}, \mathrm{G})=\inf _{\mathrm{g} \in \mathrm{G}}\|\mathrm{f}-\mathrm{g}\| \text { is called the minimal deviation off from } \mathrm{G} .
$$

The set of all elements $\mathrm{g}_{0} \in \mathrm{G}$ that are called best approximation to $\mathrm{x} \in \mathrm{X}$ is
$\mathrm{P}_{\mathrm{G}}(\mathrm{x})=\left\{\mathrm{g}_{0} \in \mathrm{G}:\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\|\mathrm{x}-\mathrm{g}\|\right.$ for all $\left.\mathrm{g} \in \mathrm{G}\right\}$
Hence $\mathrm{P}_{\mathrm{G}}$ defines a mapping from X into the power set of G is called the metric projection onto G , (other names nearest point mapping, proximity map)
B. Remark 1.2[1]

The set $\mathrm{P}_{\mathrm{G}}(\mathrm{x})$ of all best approximation to $\mathrm{x} \in \mathrm{X}$ can be written as

$$
\mathrm{P}_{\mathrm{G}}(\mathrm{x})=\left\{\mathrm{g}_{0} \in \mathrm{G}:\left\|\mathrm{x}-\mathrm{g}_{0}\right\|=\mathrm{d}(\mathrm{x}, \mathrm{G})\right\}
$$

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C. Definition 1.3.[3]

A set $S$, in a linear space is convex if $s_{1}, s_{2} \in S$ implies that

$$
\lambda_{1} s_{1}+\lambda_{2} \mathrm{~s}_{2} \in \mathrm{~S}
$$

If $\lambda_{1}$ and $\lambda_{2}$ are non negative and $\lambda_{1}+\lambda_{2}=1$
If $S$ is empty or consists of one point, then it is clearly convex
D. Definition 1.4[1]

If $\mathrm{P}_{\mathrm{G}}(\mathrm{x})$ contains at least one element, then the subset G is called a proximinal set.
In other words, if $\mathrm{P}_{\mathrm{G}}(\mathrm{x}) \neq \varphi$ then G is called a proximinal set
The term proximinal set (is a combination of proximity and maximal)
E. Definition 1.5[1] (Quasi-Orthogonal Set)

Let X be a normed linear space, and $G$ a nonempty subset of X . Then we say that $\perp_{\mathrm{B}} \hat{G}$ for every $\mathrm{g} \in \mathrm{G}$.
where $\hat{G}=\{x \in X ;\|x\|=d(x, G)\}=\left\{x \in X: x \perp_{B} G\right\}$.

## F. Remark 1.6[1]

In a Hilbert space, any closed subspace is quasi-orthogonal.
Proof:
Let H be a Hilbert space and G a closed subspace of H .
Then $\hat{G}=G^{\perp}=\{y \in H:\langle x, y\rangle=0$, for all $x \in G\}$. Then $G \perp \hat{G}$.
Therefore G is quasi-orthogonal subspace of H .
G. Definition 1.7[2]

Let $X$ be a normed linear space and $G$ be a subset of $X$, and $\varepsilon>0$. A point $g_{0} \in G$ is said to be $\varepsilon$-best approximation for $\mathrm{x} \in \mathrm{X}$ if and only if
$\left\|x-g_{0}\right\| \leq\|x-\mathrm{g}\|+\varepsilon$ for all $g \in G$
H. Remark 1.8[2]

For $\mathrm{x} \in \mathrm{X}$, the set of all $\varepsilon$-Best approximation of x in G is denoted by
$\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$, in other words,
$\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)=\left\{\mathrm{g}_{0} \in \mathrm{G}:\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon\right.$ for all $\left.\mathrm{g} \in \mathrm{G}\right\}$.
I. Theorem 1.9[2]

Let $G$ be a subspace of a normed linear space $X$. Then $\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ is bounded.
Proof:
Let $\mathrm{g}_{1}, \mathrm{~g}_{2} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$, then $\left\|\mathrm{x}-\mathrm{g}_{1}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon$ for all $\mathrm{g} \in \mathrm{G}$, and

$$
\left\|\mathrm{x}-\mathrm{g}_{2}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon \text { for all } \mathrm{g} \in \mathrm{G}
$$

Now, $\left\|\mathrm{g}_{1}-\mathrm{g}_{2}\right\|=\left\|\mathrm{g}_{1}-\mathrm{x}+\mathrm{x}-\mathrm{g}_{2}\right\| \leq\left\|\mathrm{x}-\mathrm{g}_{1}\right\|+\left\|\mathrm{x}-\mathrm{g}_{2}\right\|$

$$
\leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon+\|\mathrm{x}-\mathrm{g}\|+\varepsilon=2\|\mathrm{x}-\mathrm{g}\|+2 \varepsilon=\mathrm{k},
$$

so we have $\left\|\mathrm{g}_{1}-\mathrm{g}_{2}\right\| \leq \mathrm{k}$ where $\mathrm{k}=2 \mathrm{~d}(\mathrm{x}, \mathrm{G})+2 \varepsilon$.
Therefore, $\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ is bounded.
Hence the proof

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## J. Theorem 1.10[2]

Let $G$ be a subspace of normed linear space $X$, and $x \in X$. Then $P_{G}(x, \varepsilon)$ is convex.
Proof:
Let $\mathrm{g}_{1}, \mathrm{~g}_{2} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$, and $0 \leq \lambda \leq 1$, then $\left\|\mathrm{x}-\mathrm{g}_{1}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon$ for all $\mathrm{g} \in \mathrm{G}$, and $\left\|\mathrm{x}-\mathrm{g}_{2}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon$ for all $\mathrm{g} \in \mathrm{G}$
Now, $\left\|x-\left(\lambda g_{1}+(1-\lambda) g_{2}\right)\right\|=\left\|x-\lambda g_{1}-g_{2}+\lambda g_{2}\right\|$

$$
\begin{aligned}
& =\left\|\mathrm{x}-\lambda \mathrm{g}_{1}-\mathrm{g}_{2}+\lambda \mathrm{g}_{2}+\lambda \mathrm{x}-\lambda \mathrm{x}\right\| \\
& =\left\|\lambda\left(\mathrm{x}-\mathrm{g}_{1}\right)+(1-\lambda)\left(\mathrm{x}-\mathrm{g}_{2}\right)\right\| \\
& \leq\left\|\mathrm{x}-\mathrm{g}_{1}\right\|+(1-\lambda)\left\|\mathrm{x}-\mathrm{g}_{2}\right\| \\
& \leq \lambda(\|\mathrm{x}-\mathrm{g}\|+\varepsilon)+(1-\lambda)\|\mathrm{x}-\mathrm{g}\|+\varepsilon) \\
& =\|\mathrm{x}-\mathrm{g}\|+\varepsilon .
\end{aligned}
$$

Thus, $\lambda \mathrm{g}_{1}+(1-\lambda) \mathrm{g}_{2} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$.
Hence $\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ is convex.
Hence the proof

## K. Definition 1.11.[2] ( $\varepsilon$-orthogonality)

Let X be a normed linear space, $\varepsilon>0$, and $\mathrm{x}, \mathrm{y} \in \mathrm{X}$. We call x is $\varepsilon$ - orthogonal to y and is denoted by $\mathrm{x} \perp_{\varepsilon} \mathrm{y}$ if and only if $\|\mathrm{x}+\alpha \mathrm{y}\|+\varepsilon \geq\|\mathrm{x}\|$ for all scaler $\alpha$ with $|\alpha| \leq 1$
For subsets $G_{1}, G_{2}$ of $X, G_{1} \perp_{\varepsilon} G_{2}$ if and only if, $g_{1} \perp_{\varepsilon} g_{2}$ for all $g_{1} \in G_{1}, g_{2} \in G_{2}$.
L. Theorem: 1.12[2]

Let $X$ be a normed linear space, $G$ be a subspace of $X$, and $\varepsilon>0$. Then for all $X \in X$, $\mathrm{g}_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ if and only if $\left(\mathrm{x}-\mathrm{g}_{0}\right) \perp_{\varepsilon} \mathrm{G}$.
Proof:
( $\Rightarrow>$ ) Suppose $g_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$. Put $\mathrm{g}_{1}=\mathrm{g}_{0}-\alpha \mathrm{g}$ for $\mathrm{g} \in \mathrm{G}$ and $|\alpha| \leq 1$.
Since $\mathrm{g}_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ and $\mathrm{g}_{1} \in \mathrm{G}$ so, then, $\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\left\|\mathrm{x}-\mathrm{g}_{1}\right\|+\varepsilon$, then
$\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\left\|\mathrm{x}-\left(\mathrm{g}_{0}-\alpha \mathrm{g}\right)\right\|+\varepsilon$, and this implies that
$\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\left\|\left(\mathrm{x}-\mathrm{g}_{0}\right)+\alpha \mathrm{g}\right\|+\varepsilon$.
Therefore, $\left(x-g_{0}\right) \perp_{\varepsilon} G$.
$(<=)$ Let $\left(\mathrm{x}-\mathrm{g}_{0}\right) \perp_{\varepsilon} \mathrm{G}$, then for all $\alpha$ with $|\alpha| \leq 1$ and $\mathrm{g}_{1} \in \mathrm{G}$
we have,
$\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\left\|\mathrm{x}-\mathrm{g}_{0}+\alpha \mathrm{g}_{1}\right\|+\varepsilon$
For any $g \in G$ by putting $g_{1}=g_{0}-g$ and $\alpha=1$, the last inequality implies,
$\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon$
Therefore, $\mathrm{g}_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$
Hence the proof
M. Notation 1.13

Let $X$ be a normed linear space, and $G$ a subspace of $X$, and for $\varepsilon>0$, let
$\mathrm{P}_{\mathrm{G}}{ }^{-1}(0, \varepsilon)=\{\mathrm{x} \in \mathrm{X}:\|\mathrm{x}\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon$ for all $\mathrm{g} \in \mathrm{G}\}=\left\{\mathrm{x} \in \mathrm{X}: \mathrm{x} \perp_{\varepsilon} \mathrm{G}\right\}$
Then, $\hat{\mathrm{G}}_{\varepsilon}=\left\{\mathrm{x} \in \mathrm{X}: \mathrm{x} \perp_{\varepsilon} \mathrm{G}\right\}$.
N. Lemma 1.14[2]

Let G be a subspace of a normed linear space X . Then for all $\mathrm{x} \in \mathrm{X}$ and all $\varepsilon>0$, we have, $\mathrm{g}_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ if and only if $\left(\mathrm{x}-\mathrm{g}_{0}\right) \in \hat{\mathrm{G}}_{\varepsilon}$
Proof:
$\mathrm{g}_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ if and only if by [Theorem1.12], $\left(\mathrm{x}-\mathrm{g}_{0}\right) \perp_{\varepsilon} \hat{\mathrm{G}}_{\varepsilon}$ if and only if $\left(\mathrm{x}-\mathrm{g}_{0}\right) \in \hat{\mathrm{G}}_{\varepsilon}$.
O. Corollary 1.15

Let $G$ be a subspace of a normed linear space $X$, and let $\varepsilon>0, x \in X$. Then,
$\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)=\mathrm{G} \cap\left(\mathrm{x}-\hat{\mathrm{G}}_{\varepsilon}\right)$
Proof:
$\mathrm{g}_{0} \in \mathrm{G} \cap\left(\mathrm{x}-\hat{\mathrm{G}}_{\varepsilon}\right)$ if and only if $\mathrm{g}_{0} \in \mathrm{G}$, and $\mathrm{g}_{0} \in\left(\mathrm{x}-\hat{\mathrm{G}}_{\varepsilon}\right)$ if and only if $\mathrm{g}_{0} \in \mathrm{G}$ and $\mathrm{g}_{0}=\mathrm{x}-\hat{\mathrm{g}}$, where $\hat{\mathrm{g}} \in \hat{\mathrm{G}}_{\varepsilon}$ if and only if $\mathrm{g}_{0} \in \mathrm{G}, \hat{\mathrm{g}}=\left(\mathrm{x}-\mathrm{g}_{0}\right) \in \hat{\mathrm{G}}_{\varepsilon}$ if and only if $\mathrm{g}_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$ by [ Lemma 1.14].
Therefore, $\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)=\mathrm{G} \cap\left(\mathrm{x}-\hat{\mathrm{G}}_{\varepsilon}\right)$
Hence the proof
P. Theorem 1.16

Let $G$ be a subspace of a normed linear space $X, \varepsilon>0$, and $\varepsilon \geq \alpha$. Then, $\hat{\mathrm{G}} \subseteq \hat{\mathrm{G}}_{\alpha} \subseteq \hat{\mathrm{G}}_{\varepsilon}$, and therefore $\cap_{\varepsilon>0} \hat{\mathrm{G}}_{\varepsilon}=\hat{\mathrm{G}}$
Proof:
Let $x \in \hat{G}$, then $\|x\| \leq\|x-g\|$ for all $g \in G$.
Now $\|\mathrm{x}\| \leq\|\mathrm{x}-\mathrm{g}\| \leq\|\mathrm{x}-\mathrm{g}\|+\alpha[\alpha>0]$, so, we have $\mathrm{x} \in \hat{\mathrm{G}}_{\alpha}$.
Hence $\hat{\mathrm{G}} \subset \hat{\mathrm{G}}_{\alpha}$ $\qquad$
Let $\mathrm{x} \in \hat{\mathrm{G}}_{\alpha}$, then $\|\mathrm{x}\| \leq\|\mathrm{x}-\mathrm{g}\|+\alpha \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon[\varepsilon>\alpha]$, this implies that $\mathrm{x} \in \hat{\mathrm{G}}_{\varepsilon}$, and so, $\hat{\mathrm{G}}_{\alpha} \subset \hat{\mathrm{G}}_{\varepsilon}$
(1) and (2) together imply that $\hat{\mathrm{G}} \subset \hat{\mathrm{G}}_{\alpha} \subset \hat{\mathrm{G}}_{\varepsilon}$,

Now, we show $\cap_{\varepsilon>0} \hat{\mathrm{G}}_{\varepsilon}=\hat{\mathrm{G}}$
From above we have $\hat{\mathrm{G}} \mathrm{C} \cap_{\varepsilon>0} \hat{\mathrm{G}}_{\varepsilon}$
conversely, let $\mathrm{x} \in \cap_{\varepsilon>0} \hat{\mathrm{G}}_{\varepsilon}$,
Then for all $\varepsilon>0,0 \leq\|x\| \leq\|x-g\|+\varepsilon$ for all $g \in G$, then for all $n \in N$,
$0 \leq\|x\| \leq\|x-g\|+\frac{1}{n}$ for all $g \in G$ :
As $n \rightarrow \infty,\|x\| \leq\|x-g\|$ for all $g \in G$, then $x \in \hat{G}$,
and so,
$\cap_{\varepsilon>0} \hat{\mathrm{G}}_{\varepsilon} \underline{C} \hat{\mathrm{G}}$
Therefore $\cap_{\varepsilon>0} \hat{\mathrm{G}}_{\varepsilon}=\hat{\mathrm{G}}$
Hence the proof.
Q. Lemma 1.17

Let G be a subspace of a normed linear space X . Then.

1) If $\varepsilon>0, \mathrm{x}, \mathrm{g} \in \mathrm{X}$ and $\mathrm{x} \perp_{\varepsilon} \mathrm{g}$, then $\mathrm{x} \perp_{\delta} \mathrm{g}$ for all $\delta \geq \varepsilon$.
2) If $x, g \in X$ and $x \perp_{B} g$, then $x \perp_{\varepsilon} g$ for all $\varepsilon>0$.
3) If $x \in X$, and $\varepsilon>0$, then $0 \perp_{\varepsilon} x, x \perp_{\varepsilon} 0$.
4) If $x \perp_{\varepsilon} g$ and $|\beta|<1$, then $\beta x \perp_{\varepsilon} \beta$ g.

Proof:
(a) Let $\varepsilon>0, \mathrm{x}, \mathrm{g} \in \mathrm{X}$ and $\mathrm{x} \perp_{\varepsilon} \mathrm{g}$, then by [Definition 1.11] we have $\|\mathrm{x}\| \leq\|\mathrm{x}+\alpha \mathrm{g}\|+\varepsilon$, where $|\alpha| \leq 1$ and $\varepsilon>0$
Then, $\|\mathrm{x}\| \leq\|\mathrm{x}+\alpha \mathrm{g}\|+\varepsilon \leq\|\mathrm{x}+\alpha \mathrm{g}\|+\delta$, [since $\delta \geq \varepsilon$ ]
Therefore, $\mathrm{x} \perp_{\delta} \mathrm{g}$
(b) Let $\mathrm{x}, \mathrm{g} \in \mathrm{X}$ and $\mathrm{x} \perp_{\mathrm{B}} \mathrm{g}$, then $\|\mathrm{x}\| \leq\|\mathrm{x}+\alpha \mathrm{g}\|$ for all $\alpha \in \mathbb{R}$

Since $\varepsilon>0$, then $\|\mathrm{x}\| \leq\|\mathrm{x}+\alpha \mathrm{g}\| \leq\|\mathrm{x}+\alpha \mathrm{g}\|+\varepsilon$ for all $|\alpha| \leq 1$
Hence $\mathrm{x} \perp_{\varepsilon}$ g for all $\varepsilon>0$
(c) Let $\mathrm{x} \in \mathrm{X}$ and $\varepsilon>0$, then $\|0\| \leq\|0+\alpha \mathrm{x}\|+\varepsilon$, and so $0 \perp_{\varepsilon} \mathrm{X}$.

We have also $\|x\| \leq\|x\|+\varepsilon$, then $\|x\| \leq\|x+\alpha 0\|+\varepsilon$, Hence $\mathrm{x} \perp_{\varepsilon} 0$.
(d) Let $\mathrm{x} \perp_{\varepsilon} \mathrm{g}$, and $|\beta|<1$, then $\|\mathrm{x}\| \leq\|\mathrm{x}+\alpha \mathrm{g}\|+\varepsilon$.

Multiply both sides by $|\beta|$,
we get $|\beta|\|x\| \leq\|\beta x+\beta \alpha \mathrm{g}\|+|\beta| \varepsilon$
$\leq\left\|\beta x+\alpha_{1} g\right\|+\varepsilon$, and so
$\|\beta \mathrm{x}\| \leq\left\|\beta \mathrm{x}+\alpha_{1} \mathrm{~g}\right\|+\varepsilon$
Therefore, $\beta \mathrm{x} \perp_{\varepsilon} \beta \mathrm{g}$
Hence the proof

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## R. Theorem 1.18

Let $G$ be a subspace of a normed linear space $X$. If $x \in X, \varepsilon>0$
and $\delta \geq \varepsilon$, then $\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon) \subseteq \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \delta)$.
Proof:
Let $g_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon)$. Then by [Definition 1.7], we have
$\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon$ for all $\mathrm{g} \in \mathrm{G}$ and $\varepsilon>0$
Then $\left\|\mathrm{x}-\mathrm{g}_{0}\right\| \leq\|\mathrm{x}-\mathrm{g}\|+\varepsilon \leq\|\mathrm{x}-\mathrm{g}\|+\delta$
[since $\delta>\varepsilon$ ], then, $g_{0} \in \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \delta)$.
Therefore $\mathrm{P}_{\mathrm{G}}(\mathrm{x}, \varepsilon) \subseteq \mathrm{P}_{\mathrm{G}}(\mathrm{x}, \delta)$
Hence the proof

## II. CONCLUSION

Here, I conclude my paper as $\varepsilon$-Best approximation and $\varepsilon$ - orthogonality has the properties which are similar to the properties of best approximation.

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