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The Structure of GK Algebras

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Abstract: In this paper, the new notion which is called GK algebra from a non-empty set is introduced. The basic properties of GK algebra are analyzed.
Keywords: GK algebra, commutative, associative, self-distributive, subalgebra.

I. INTRODUCTION

In 1966, the concept of BCK and BCI algebras are introduced by Iseki [3]. Since Kim and Yon [8] studied on dual BCK algebras and MV algebra, it is known that BCK algebras is a proper subclass of BCI algebras. The concept of BE algebra which is a generalization dual BCK was introduced by Kim and Y.H. Kim [7]. Meng [9] introduced the concept of CI algebra as a generalization of BE algebra and also discussed about some of its properties and relations with BE algebras.

II. PRELIMINARIES

A. Definition:2.1 [7] An algebra $(X,*,1)$ of type (2,0) is said to be a BE-algebra if it satisfies the following
1) $x*x=1$
2) $x*1=1$
3) $1*x=x$
4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$

B. Definition:2.2 [9] A CI-algebra is an algebra $(X,*,1)$ of type (2,0) satisfying the following axioms
1) $x*x=1$
2) $1*x=x$
3) $x*(y*z) = y*(x*z)$ for all $x,y,z \in X$

C. Proposition:2.3 [7] If $(X,*,1)$ is a BE-algebra, then $x*(y*x) = 1$
D. Definition:2.4 [7] A BE-algebra $(X,*,1)$ is said to be self distributive if $x*(y*z)=(x*y)*(x*z)$ for all $x, y, z \in X$.
E. Proposition:2.5 [9] Any CI algebra X satisfies the condition $y*((y*x)*x)=1$ for any $x, y \in X$,

III. THE NOTION AND ELEMENTARY PROPERTIES OF GK ALGEBRA.

A. Definition:3.1
A non-empty set X with fixed constant 1 and a binary operation $*$ is called GK-algebra if it satisfying the following axioms
$x*x=1$
(ii) $x*1=x$
(iii) $x*y=1$ and $y*x=1$ implies $x=y$
(iv) $(y*z)*(x*z)=y*x$
(v) $(x*y)*(1*y)=x$ for all $x, y, z \in X$

B. Example:3.2
Consider the set $X=\{1,2,3\}$. The binary operation $*$ is defined as follows

Table:1

*	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

$\therefore (X,*,1)$ is a GK-algebra.

C. Remark:3.3

1) A GK algebra need not be a BE algebra, for $2*1=2\neq1, 3*1=3\neq1$.
2) A GK algebra need not be a CI algebra, for $1*3=2\neq3, 1*2=3\neq2$.
3) A GK algebra is said to be a CI algebra if it satisfies the additional relations,
$1*x = x$ and $x*(y*z)=y*(x*z)$
4) A GK algebra is said to be a BE algebra if it satisfies the additional relations,
$x*1 = 1$, $1*x = x$ and $x*(y*z) = y*(x*z)$.

D. Theorem:3.4 Let $(X,*,1)$ be a GK-algebra. Then

1) $1*(1*x) = x$
2) $(x*y)*1 = (x*1)*(y*1)$
3) $y*(1*(1*y)) = 1$
4) If $1*x = 1*y$ then $x = y$ for any $x,y\in X$
5) $x*(1*x))*x = x$ for any $x\in X$
6) $x*(x*y) = x = y = y*(x*x)$ for any $x,y\in X$
7) $x * (y*x) = x = y = y*(x*x)$ for any $x,y\in X$
8) $1*(x*y)=y*x$

Proof:

a) In axiom (v) $(x*y)*(1*y)=x$ of GK-algebra,
replacing y by x,
we have $(x*x)*(1*x) = x$
$\Rightarrow 1*(1*x) = x$ by axiom (i) of definition:3.1
b) By axiom (ii) $x*1=1$ of GK algebra
we have $(x*y)*1 = x*y$
$=(x*1)*(y*1)$ by axiom(ii) of definition:3.1
c) In theorem 3.4 (i), we have $1*(1*x) = x$
Now $y*(1*(1*y)) = y*y = 1$ by axiom (i) of definition:3.1
d) Let $1*x = 1*y$
Now $x = 1*(1*x)$ by theorem 3.4 (i)
$=1*(1*y)$ since $1*x = 1*y$
$=y$ by theorem 3.4 (i)
e) $(x*(1*x))*x = (x*(1*x))*(1*(1*x))$ by theorem 3.4 (i)
$=x$ by axiom (v) of definition 3.1
f) $x*(x*y) = x*1$ by axiom (iii) of definition 3.1
$= x$ by axiom (ii) of definition 3.1
$= y$ by axiom (iii) of definition 3.1
$= y*1$ by axiom (i) of definition 3.1
$= y*(x*x)$ by axiom (i) of definition 3.1
g) The proof is similar to previous proof of (vi).
h) In axiom (iv) $(y*z)*(x*z)=y*x$ of GK algebra,
replacing z by y, we have $(y*y)*(x*y)=y*x$
$\Rightarrow 1*(x*y)=y*x$ by axiom (i) of definition 3.1 .

E. Theorem:3.5

Left and Right cancellation law holds in GK-algebra

1) Right cancellation law : if $x*y = z*y$ then $x = z$
2) Left cancellation law : if $z*x = z*y$ then $x = y$

Proof:

a) Let us assume that $x*y = z*y$

Then, $x = (x*y)*(1*y)$ by axiom (v) of definition 3.1

$= (z*y)*(1*y)$

$= z$ by axiom (v) of definit ion 3.1

b) Assume that $z*x = z*y$

Now, $z*(z*x) = x*(z*z)$ by theorem 3.4 (vi)

$= x*1$ by axiom (i) of definition 3.1

$= x$ by axiom (ii) of definition 3.1

and, $z*(z*y) = y*(z*z)$ by theorem 3.4 (vi)

$= y*1$ by axiom (i) of definition 3.1

$= y$ by axiom (ii) of definition 3.1

Since $z*x = z*y$ implies $x = y$.

F. Theorem:3.6

Let $(X,*,1)$ be a group with respect to $x*y = xy^{-1}$, then $(X,*,1)$ is a GK algebra.

Proof:

We see that $x*x = xx^{-1}=1$

and $x*1 = x1^{-1} = x$

For any $x,y \in X$, we have $x*y = xy^{-1}$

when $x = y$, then $x*y= xy^{-1} =xx^{-1} =1 =yy^{-1} =yx^{-1} = y*x$.

For any $x,y,z \in X$,we have $(y*z)*(x*z) = (yz^{-1})(xz^{-1})^{-1}$

$=(yz^{-1})(zx^{-1})$

$=y(zz^{-1})x^{-1}$

$=yx^{-1}$

$=y*x$

For any $x,y \in X$, $(x*y)*(1*y) = (xy^{-1})(1y^{-1})^{-1}$

$=(xy^{-1})(y)$

$=x(y^{-1}y)$

$=x$

Hence $(X,*,1)$ is a GK-algebra.

G. Definition:3.7

A GK algebra X is said to be associative if it satisfies $(x*y)*z = x*(y*z)$ for all $x,y,z \in X$. **Theorem:3.8** Every Gk algebra $(X,*,1)$ satisfying the associative law is group under the operation "$*$".

Proof:

Putting $x = y = z$ in the associative law $(x*y)*z = x*(y*z)$

we have $(x*x)*x = x*(x*x)$

$\Rightarrow 1*x = x*1$ by axiom (i) of definition 3.1

$=x$ by axiom (ii) of definition 3.1

$\Rightarrow 1*x=x*1=x$

This means that 1 acts as the identity element in X. By axiom (i) of definition 3.1, every element x of X has its own inverse.

Now, $(y*z)*(x*z) = y*(z*(x*z))$

$= y*(x*(z*z))$

$= y*x$

and $(x*y)*(1*y) = x*(y*(1*y))$

$= x*((y*1)*y)$

$= x*(y*y)$

$=x*1$

$= x$.

Therefore $(X,*,1)$ is a group.

H. Definition:3.9

A GK-algebra $(X,*,1)$ is a self-distributive if the operation $*$ is

1) Right distributive law $(x*y)*z = (x*z)*(y*z)$ for all $x,y,z \in X$.
2) Left distributive law $x*(y*z) = (x*y)*(x*z)$ for all $x,y,z \in X$.

I. Definition:3.10

A GK algebra X is said to be commutative if it satisfies for all $x,y \in X$, $(x*y)*y = (y*x)*x$.

J. Proposition:3.11

Let X be a GK algebra. If $x \neq y$ and $x * y = 1$ then $y*x \neq 1$.

K. Proposition:3.12

Let $(X,*,1)$ be a GK algebra. Then for any $x,y,z \in X$,

1) $x*(x*(y*x)) = 1$
2) $y*(y*(x*y)) = 1$
3) $(x*y)*x = (y*x)*y$
4) $(x*y)*y = (y*x)*x$
5) $(x*y)*x = (x*x)*y$
6) $(x*y)*y = (y*y)*x$

Proof:

a) Let us consider $x*(x*(y*x))$
$= x*(x*1)$ by axiom (iii) of definition 3.1
$= x*x$ by axiom (ii) of definition 3.1
$= 1$ by axiom (i) of definition 3.1.
b) The proof is similar to proof (i) in proposition 3.12 .
c) Consider $(x*y)*x$
$=1*x$
$=1*y$ by axiom (iii) of definition 3.1
$=(x*y)*y$ by axiom (iii) of definition 3.1
$=(y*x)*y$ by axiom (iii) of definition 3.1
d) Consider $(x*y)*y$
$=1*y$
$=1*x$ by axiom (iii) of definition 3.1
$=(x*y)*x$ by axiom (iii) of definition 3.1
$=(y*x)*x$ by axiom (iii) of definition 3.1

This proof shows that the commutativity of GK algebra.

(v) Consider $(x*y)*x = 1*x = 1*y = (x*x)*y$ by axiom(i) & (iii) of definition 3.1

(vi) Proof is similar to (v) in proposition 3.12.

L. Theorem:3.13

In GK algebra X, for any $x,y,z \in X$ if associativity holds then the following are equivalent

1) $x*(y*z) = (x*z)*y$
2) $(y*z)*(x*z) = y*x$

Proof

(i)⇒(ii) Assume $x*(y*z) = (x*z)*y$

Then $(y*z)*(x*z) = ((y*z)*z)*x$
$=(y*(z*z))*x$
$=(y*1)*x$ by axiom (i) of definition 3.1

$=y*x$ by axiom (ii) of definition 3.1

(ii)⇒(i) Assume $(y*z)*(x*z)=y*x$

Then $x*(y*z) = (x*z)*((y*z)*z)$

$= (x*z)*(y*(z*z))$

$= (x*z)*(y*1)$ by axiom (i) of definition 3.1

$= (x*z)*y.$ by axiom (ii) of definition 3.1

M. Definition:3.14

Let $(X,*,1)$ be a GK-algebra. A non-empty subset A of X is called a subalgebra of X if

$x*y \in A$ for any $x,y \in A$.

N. Theorem:3.15

Let $(X,*,1)$ be a GK algebra and $A \neq \varphi$, $A \subseteq X$ then the following are equivalent

1) A is a subalgebra of X
2) $x*(1*y)$, $1*y \in A$ for any $x,y \in A$

Proof:

(i)⇒(ii) Let A be a subalgebra of X. Since A is a subset of X which is non-empty there exists an element $x \in A$ such that $x*x = 1 \in A$.

Since X is closed under '$*$', $y \in A$, $1*y \in A \Rightarrow x*(1*y) \in A$.

(ii)⇒(i) Since $x*y = x*(1*(1*y))$ by theorem 3.4 (i)

which implies $x*y \in A$ for any $x,y \in A$.

∴ A is a subalgebra of X.

IV. CONCLUSION

In this paper the notion of GK algebra is introduced and studied about some of their properties. It may lead our future study of GK algebra such as homomorphism of GK algebra, filter of GK algebra and Ideal theory on GK algebra.

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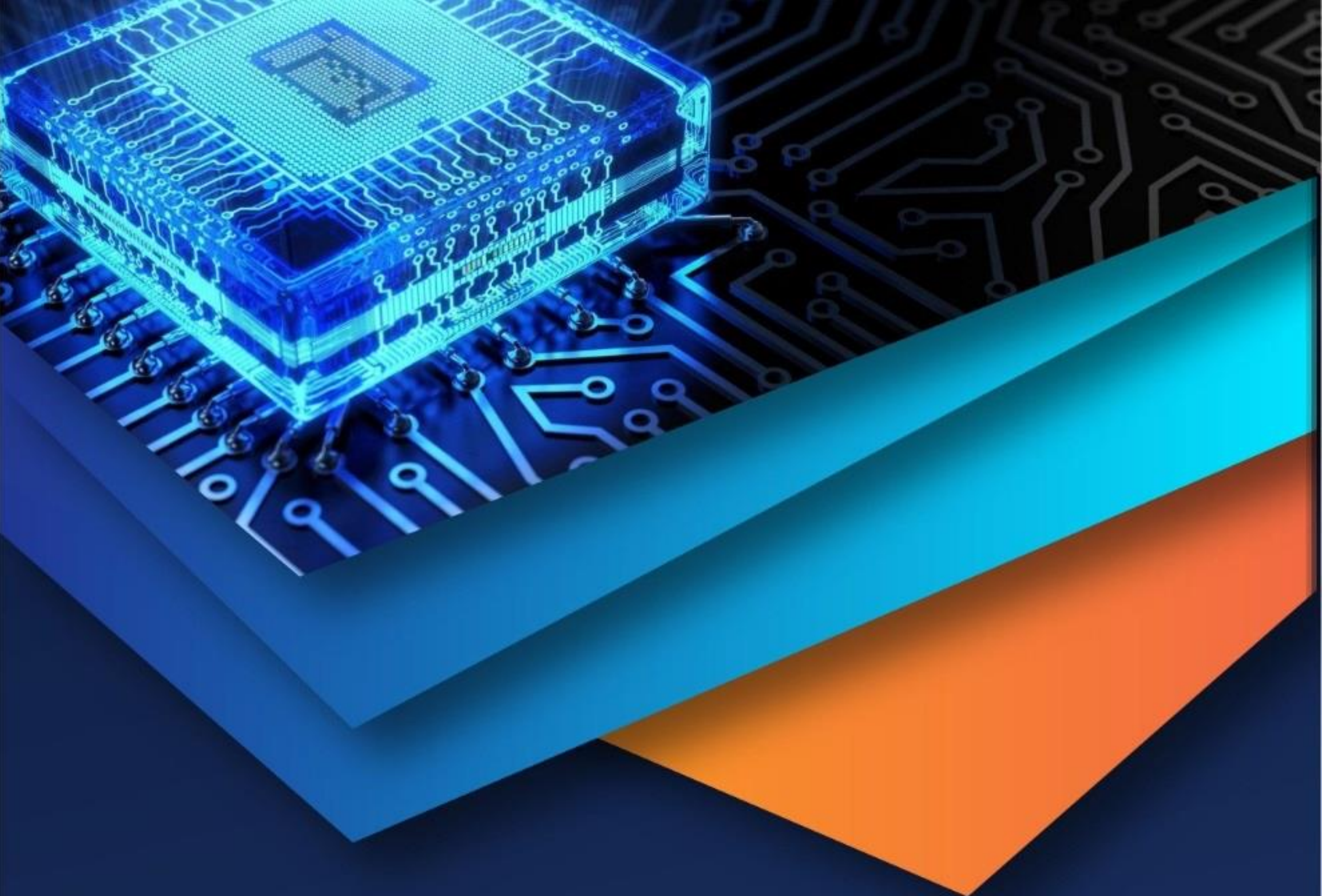

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