iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 6 **Issue:** IV **Month of publication:** April 2018

DOI: http://doi.org/10.22214/ijraset.2018.4452

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 6 Issue IV, April 2018- Available at www.ijraset.com

Solving Boundary Value Problems in Ordinary Differential Equations by using Maclaurin Series

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Abstract: In this paper we explained a new powerful technique to find the solution of boundary value problems in ordinary differential equation. Where in we used Maclaurin series to find the analytical solution of BVP's[3]. This method can be effectively used in solving the BVP's with different types of boundary conditions. We succeeded to find the optimal solution in this approach by taking different order and types of conditions [2].
Keywords: Ordinary differential equations, Boundary value problem, Maclaurin's series.

I. INTRODUCTION

Aim of our work is to find a new and simple method to find solution for boundary value problems in ordinary differential equations. It is well known that the solution of differential equation is the solution of many problems in science and engineering fields. Our method may be helpful in future to solve many problems of microscopic world as we know that first and second order linear differential equations with boundary conditions have enormous number of applications in physics, chemistry, fluid mechanics, medical science, magneto hydrodynamics, economics etc. Therefore, Maclaurin series solution may become a tool to solve many problems of above mentioned fields. This solution is first of its kind in the literature and hence in this direction our work may be helpful for solving some unsolved problems. Here we tried to explain new method by taking four different types of example [2]. We have taken y as dependent variable and x as independent variable throughout this article. First let us take a simple first order differential equation with Dirichlet Boundary Conditions.

Ex.1 Solve $y'+y=x$ with $y(0)=-1$.

Solution:

Given equation is $y'+y=x$ (1)

with $y(0)=-1$ (2)

We assume the solution as the powers of independent variables, as in power series method [2, 3, 4]. Let us assume the solution as Maclaurin series as below,

$y = a + bx+cx^2+d\,x^3+ ex^4+ fx^5$.............. (3)

Differentiating (3) w.r.t x

$y'=b+2cx+3\,dx^2+4\,ex^3+5f\,x^4$ .... (4)

Substituting (3) in (2) and put x=0,

$a=-1$ (5)

Substituting (3) and (4) in (1)

That is $y'+y=x$

$\Rightarrow b+2cx+3\,dx^2+4\,ex^3+5f\,x^4$ +.......$a+bx+cx^2+d\,x^3+ ex^4+ fx^5$.........= x

Comparing coefficients of power terms in LHS and RHS

$a+b=0\Rightarrow -1+b=0\Rightarrow b = 1$ $\because a=-1$

$2c+b=1\Rightarrow 2c+1=1\Rightarrow c=0$ $\because b=1$

$3d+c=0\Rightarrow d=0$ $\because c=0$

....

....

and so on

Substituting the values of a,b,c,d.......in (3),

$y=-1+1.x+0+0+$...........

$\therefore$ y=x-1 is the solution of the given differential equation(1).

So, if we have this type of problems with simple boundary condition we can apply Maclaurin Series method and find solution very easily. This approach can be used in any physical problems involving first order differential with Dirichlet boundary conditions in similar way.

Now, we consider, BVP of second order with Dirichlet Boundary Conditions at two points.

Ex:2 Solve $y'' - y = x$ with $y(0)=y(1)=0$[2]

Solution:

Given equation is $y'' - y = x$ (1)

with $y(0)=0$ and $y(1)=0$ (2)

Assume,

$y = a+bx+cx^2+dx^3+ex^4+fx^5$............. (3)

Differentiating (3) w.r.t x

$y'=b+2cx+3dx^2+4ex^3+5fx^4$ .... (4)

Differentiating (4) w.r.t x

$y''=2c+6dx+12ex^2+20fx^3$ .... (5)

from (2) and (3) we get

$a=0$ (6)

Substituting (3) and (5) in (1),

$2c+6dx+12ex^2+20fx^3$ ....-[$a+bx+cx^2+dx^3+ex^4+fx^5$.............] = x

$\Rightarrow(2c-a)+(6d-b)x+(12e-c)x^2+(20f-d)x^3 \dots\dots\dots\dots\dots\dots\dots = x$

Comparing coefficients of x powers in LHS and RHS,

$2c-a = 0 \Rightarrow c=0$ $\because a=0$

$6d-b = 1 \Rightarrow b= 6d - 1$

$12e-c = 0 \Rightarrow e = 0$ $\because c=0$

$20f-d = 0 \Rightarrow f= \frac{d}{20}$

....

....

And so on

Substituting the value of a,b,c,d,e,f....... in (3)

$y = 0+(6d-1)x+0+dx^3+0+\frac{d}{20}x^5+$...........

$\Rightarrow y = 6dx - x + dx^3+\frac{d}{20}x^5 \dots\dots\dots\dots$

$\Rightarrow y = -x + 6d[x+\frac{1}{6}x^3+\frac{1}{20}x^5 \dots\dots\dots\dots]$

$\Rightarrow y = -x + 6d[x+ \frac{1}{3!}x^3+\frac{1}{5!}x^5 \dots\dots\dots\dots]$

$\therefore y=-x+6d(\sinh(x))$ (7)

From (2) $y(1) = 0$,

(7) $\Rightarrow y(1)=-1+6d(\sinh(1))$

$\Rightarrow 0 = -1+6d(\sinh(1)) \Rightarrow d = \frac{1}{6(\sinh(1))}$

Substituting the value of d in (7),

$\therefore y=-x+\frac{(\sinh(x))}{(\sinh(1))}$ or

$y=\frac{(\sinh(x))}{(\sinh(1))} - x$ is the solution of the given differential equation(1).

(The same problem is done in[2] by using Greens Function)

In this case also we get the solution by using Maclaurin's series method. This method can be used in any model were we need solve second order differential equation with Dirichlet Boundary Conditions at two points.

The same technique can be used to find the solution of boundary value problem with mixed boundary conditions, that is Robin boundary condition , Neumann and Dirichlet Boundary Conditions.

Ex:3 Solve $y'' - y = -2e^x$ with $y(0)=y'(0)$ and $y(l)+y'(l)=0$

Solution:

Given equation is $y'' - y = -2e^x$ (1)

with $y(0)=y'(0)$ (2)

and

$y(l) + y'(l) = 0$ (3)

$y = a+bx+cx^2+dx^3+ex^4+fx^5+gx^6$............ (4)

Differentiating (4) w.r.t x

$y'=b+2cx+3dx^2+4ex^3+5fx^4+6gx^5$ .... (5)

Differentiating (5) w.r.t x

$y''=2c+6dx+12ex^2+20fx^3+30gx^4$ .... (6)

form equations (2) , (4) and (5),

$a = b$ (7)

Substituting equations (4) and (6) in (1),

$2c+6dx+12ex^2+20fx^3+30gx^4$ .....$-[a+bx+cx^2+dx^3+ex^4+fx^5$......]$= -2e^x$

$\Rightarrow (2c-a)+(6d-b)x+(12e-c)x^2+(20f-d)x^3+(30g-e)x^4+\ldots$

$=-2\{1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5\ldots\ldots\}$

Comparing the coefficients of x powers in LHS and RHS,

$2c - a = -2 \Rightarrow 2c - b = -2 \Rightarrow c = \frac{b}{2} - 1$ $\quad\because a = b$

$6d - b = -2 \Rightarrow d = \frac{b}{6} - \frac{1}{3}$

$12e - c = -1 \Rightarrow e = \frac{c}{12} - \frac{1}{12} = \frac{1}{12}[\frac{b}{2} - 1]\ \frac{1}{12} = \frac{b}{24} - \frac{1}{6}$ $\quad\because c = \frac{b}{2} - 1$

$20f - d = \frac{-1}{3} \Rightarrow f = \frac{1}{20}[\frac{b}{6} - \frac{1}{3}] - \frac{1}{60} = \frac{b}{120} - \frac{1}{30}$ $\quad\because d = \frac{b}{6} - \frac{1}{3}$

$30g - e = \frac{-1}{12} \Rightarrow g = \frac{e-12}{30} = \frac{b}{30*24} - \frac{1}{120}$

....

....

And so on.

Substituting (4) and (5) in (3) and put x=l,

$b+bl+cl^2+dl^3+el^4+fl^5+gl^6+\ldots+b+2cl+3dl^2+4el^3+5fl^4+6gl^5\ldots=0$

$2b+(b+2c)l+(c+3d)l^2+(d+4e)l^3+(e+5f)l^4+(f+6g)l^5+\ldots\ldots\ =0$

now, substituting the values of c, d, e, f, g..... in the above equation,

$\Rightarrow 2b+[b+2(\frac{b}{2}-1)]l+[(\frac{b}{2}-1)+3(\frac{b}{6}-\frac{1}{3})]l^2+[(\frac{b}{6}-\frac{1}{3})+4(\frac{b}{24}-\frac{1}{6})]l^3+[(\frac{b}{24}-\frac{1}{6})+5(\frac{b}{120}-\frac{1}{30})]l^4+[(\frac{b}{120}-\frac{1}{30})+6(\frac{b}{30*24}-\frac{1}{120})]l^5+\ldots\ =0$

$\Rightarrow 2b+(b+b-2)l+(\frac{b}{2}-1+\frac{b}{2}-1)l^2+(\frac{b}{6}-\frac{1}{3}+\frac{b}{6}-\frac{2}{3})l^3+(\frac{b}{24}-\frac{1}{6}+\frac{b}{24}-\frac{1}{6})l^4$

$+\ldots\ldots\ = 0$

$\Rightarrow 2b+2bl+2bl^2+\frac{b}{3}l^3+\frac{b}{12}l^4+\ldots = 2l+2l^2+l^3+\frac{1}{3}l^4\ldots$

$\Rightarrow 2b[1+l+\frac{l^2}{2!}+\frac{l^3}{3!}+\frac{l^4}{4!}\ \ldots\ldots\ldots] = 2l[1+l+\frac{l^2}{2!}+\frac{l^3}{3!}+\frac{l^4}{4!}+\cdots]$

$\Rightarrow 2be^l=2le^l$

$\Rightarrow b = l$

Now , substituting b = l in b, c, d, e, f, g,........... and then substituting substituted value in (4),

$$y = l + lx + \left(\frac{l}{2} - 1\right)x^2 + \left(\frac{l}{6} - \frac{1}{3}\right)x^3 + \left(\frac{l}{24} - \frac{1}{6}\right)x^4 + \left(\frac{l}{120} - \frac{1}{30}\right)x^5$$

$$+\left(\frac{l}{30*24} - \frac{1}{120}\right)x^6 \ldots\ldots\ldots\ldots$$

$$\Rightarrow y = l + lx + \frac{l}{2}x^2 - x^2 + \frac{l}{6}x^3 - \frac{1}{3}x^3 + \frac{l}{24}x^4 - \frac{1}{6}x^4 + \frac{l}{120}x^5 - \frac{1}{30}x^5 + \ldots\ldots.$$

$$\Rightarrow y = l[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots] - x^2 - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 - \cdots \ldots \ldots$$

$$\Rightarrow y = le^x - x[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots] + [x + + \frac{1}{3!}x^3 + \cdots]$$

$$\Rightarrow y = le^x - xe^x + \sinh(x)$$

$\Rightarrow y = \sinh(x) + e^x(l - x)$, which is the solution for the given differential equation(1).

Thus this Maclaurin's Series method is applicable in case of mixed boundary conditions also. Now , consider an example of fourth order BVP with Cauchy Boundary Conditions.

Ex.4 Solve $y^{(iv)} = 1$, with $y(0) = y'(0) = 0$ and $y''(1) = y'''(1) = 0$

Solution:

Given equation is $y^{(iv)} = 1$ (1)

with $y(0) = y'(0) = 0$ (2)

and

$y''(1) = y'''(1) = 0$ (3)

$y = a+bx+cx^2+dx^3+ex^4+fx^5+gx^6 \ldots\ldots\ldots\ldots$ (4)

Differentiating (4) w.r.t x

$y'=b+2cx+3dx^2+4ex^3+5fx^4+6gx^5 \ldots\ldots\ldots\ldots$ (5)

Differentiating (5) w.r.t x

$y''=2c+6dx+12ex^2+20fx^3+30gx^4 \ldots\ldots\ldots\ldots$ (6)

Differentiating (6) w.r.t x

$y'''=6d+24ex+60fx^2+120gx^3 \ldots\ldots\ldots\ldots$ (7)

Differentiating (7) w.r.t x

$y^{(iv)}=24e+120fx+360gx^2 \ldots\ldots\ldots\ldots$ (8)

From (2) and (4),

a=0 (9)

from (2) and (5)

b=0 (10)

Substituting (8) in (1)

$24e+120fx+360gx^2 \ldots\ldots\ldots\ldots = 1$

Comparing the coefficients of x powers on LHS and RHS

$24e = 1 \Rightarrow e = \frac{1}{24}$

$120f = 0 \Rightarrow f = 0$

$360g=0 \Rightarrow g=0$

Now substituting (6) and (7) in (3) and put x=1,

$2c+6d+12e+20f+30g\ldots\ldots\ldots\ldots = 0$

Substituting e,f,g..... in the above equation,

2c+6d+12*$\frac{1}{24}$+0+0.......=0

2c+6d+$\frac{1}{2}$ =0 (11)

6d+$24e$+60f +120g+....=0

6d+24*$\frac{1}{24}$+0+0.. =0 ⇒d=$\frac{-1}{6}$

Substituting d=$\frac{-1}{6}$ in (11),

We get c=$\frac{1}{4}$

Now substituting a, b, c, d, e, f, g...... in (4)

y=0+0+$\frac{1}{4}x^2$ - $\frac{1}{6}x^3$ + $\frac{1}{24}x^4$

y=$\frac{1}{24}x^2[x^2-4x+6]$ is the solution for the given differential equation(1).

This method is applicable to higher order differential equations with boundary conditions also.

II. CONCLUSION

The above problems can be solved by using Greens functions [2] also, where in one can find the solution if the Greens function exist for the given boundary value problem. If it exists, then by using Greens Function we get corresponding integral equation for the given BVP, also, one has to find out the solution for the obtained integral equation. The solution thus obtained is the solution for the given BVP. But by using Maclaurin's series method, we directly find the solution, without getting solution for the integral equation; comparatively this is much simpler than by using Greens Function method. Thus we could demonstrate the solutions by using Maclaurin series method in an easy computable way by taking different types of examples which save the time and space compared to the usage of Greens function.

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