iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH
IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 2 Issue: II Month of publication: February 2014

DOI:

INTERNATIONAL JOURNAL FOR RESEARCH IN APPLIED SCIENCE AND ENGINEERING TECHNOLOGY (IJRASET)

# Oscillation Theorems for a Second Order Delay Difference Equations

Dr.P.Mohankumar[1], A.Ramesh[2]
[1](Professor of Mathematics, Aarupadaiveedu Institute of Techonology, Vinayaka Missions University, Paiyanoor, Chengalpattu, Kancheepuram Dist- 603104 , Tamilnadu, India)
[2]( Ph.D Scholar Vinayaka Missions University , Salem-636 308, Tamilnadu India)

***Abstract: The Oscillation Theorems for a Second Order Delay Difference Equations of the form***

$$\Delta^2 u_n + p_n u_{(n-\sigma(n))} = 0,\ n \in \mathbb{N}(a) \text{ ........................(1)}$$

***The method uses techniques based on signum functions. Example is inserted to illustrate the result.***

***Keywords: Delay Difference equation, signum functions, Boundedness***

***2010MSC: 39A10***

## 1. INTRODUCTION

The purpose of this paper is to present conditions for all solutions of the linear delay difference equations of the form

$$\Delta^2 u_n + p_n u_{(n-\sigma(n))} = 0,\ n \in \mathbb{N}(a) \text{ ........................(1)}$$

Where $\Delta$ is the forward difference operator i.e. $\Delta u_n = u_{n+1} - u_n$ $[\Delta^2 u_n = \Delta(\Delta u_n)]$ to be oscillatory, and to present an oscillation theorem for the more general equation

$$\Delta(r_n \Delta u_n) + p_n f(u_n, u_{g(n)}) = 0 \text{ ..........................(2)}$$

assume the condition without further mentions $\{r_n\},\{p_n\}$ positive real sequence on interval $n \in \mathbb{N}(a)$ $\{r_n\} > 0, \{p_n\} \geq 0$ $and$ $0 \leq \sigma(n) \leq m.$ The assumptions on $f$ $and$ $g$ are stated proceeding Lemma 2 in section.

Results on the growth and boundenss of non-oscillatory solutions are presented in section 3. By a solution of (1) and (2) we mean real sequence $\{u_n\}$ satisfying equation (1) for $n \in \mathbb{N}(a)$. We consider only such solutions which are non-trivial for all large $n$. A solution $\{u_n\}$ of (1) and (2) is called non-oscillatory if it is eventually positive or negative. Otherwise it is called Oscillatory.

The problem of determining oscillation theorem for second order nonlinear difference equation has been the subject of investigation in [1-3]. Among the papers dealing with this subject we refer to [4] in which oscillatory theorem of linear difference equations of second order have been established.

The purpose of this note is to give some new criteria (sufficient conditions for oscillation of all solutions of (1). The results we obtain the discrete analogues of some theorem for a non-linear differential equation of second order with delay due to katasatos [5] and staikes &petsoulat[6].

## 2. MAIN RESULTS

The following Lemma allows the use of technique introduced by Coles [1] to give a short proof of a classical oscillation Theorem for equation (3).

**Lemma 1**

*If* $p_n \geq 0, (p_n \neq 0), 0 \leq \sigma(n) \leq m$ *for all* $n \geq a$ and $\{u_n\}$ is a solutions of equation (1) that is positive, then $\Delta u_n > 0$ for all $n$ sufficiently large and there is a constant $\kappa > 0$ such that

$$\frac{u_{(n-\sigma(n))}}{u_n} \geq \kappa$$

**Proof.**

If $\{u_n\}$ is positive, then so is $u_{(n-\sigma(n))}$ and there for $\Delta^2 u_n \leq 0$. This means the $\{u_n\}$ is bounded non-oscillatory so that if $\Delta u_n \leq 0$, $\{u_n\}$ become zero again. Therefore which is contrary to the hypothesis. Hence $\Delta u_n > 0$ for all $n$ sufficiently large. Now suppose that $n_0$ is larger than the last zero of $u_{n-m}$. Then $n \geq n_0, u_{n-m} \leq u_{(n-\sigma(n))}$ and

$$\frac{u_{(n-\sigma(n))}}{u_n} \geq \frac{u_{n-m}}{u_n}$$

Let $g$ be a function whose graph is the line tangent to the graph of $\{u_n\}$ at $(n-m, u_{n-m})$ for some $n \geq n_0$; that is

$$g_s = \Delta u_{n-m}(s-t+m) + u_{n-m}$$

Since $\{u_n\}$ is bounded,

$$\frac{u_{n-m}}{u_n} \geq \frac{u_{n-m}}{g_n} = \frac{g_{n-m}}{g_n}$$

Let

$$x = \frac{-u_{n-m}}{\Delta u_{n-m} + n - m}$$

Then $g_x = 0$ and because of similar triangles we have

$$\frac{u_{n-m}}{u_n} \geq \frac{g_{n-m}}{g_n} = \frac{(x-n+m)}{(x-n)} = \frac{u_{n-m}}{[u_{n-m} + m\Delta u_{n-m}]} \quad\text{............(4)}$$

But $\Delta u_{n-m}$ is decreasing so that the last member of (4) increases to a positive limit $\kappa$ *as* $n \to \infty$ .Hence

$$\frac{u_{(n-\sigma(n))}}{u_n} \geq \frac{u_{n-m}}{u_n} \geq \kappa$$

for all $n$ sufficiently large.

**Theorem 1**

*If* $\{p_n\} \geq 0,\ 0 \leq \sigma(n) \leq m$ *for all* $n > a$ *and* $\sum_{s=a}^{\infty} p_s = \infty$, *then a*

**proof:**

If $\{u_n\}$ be a non oscillatory solution of the equation (1) and there $\Delta^2 u_n \leq 0,\ \Delta u_n \leq 0$

for sufficiently large $n$ Let $\omega_n = \dfrac{\Delta u_n}{u_n}$ . Then (1) becomes

$$\Delta\omega_n + \omega_n^{\,2} + \frac{p_n u_{(n-\sigma(n))}}{u_n} = 0$$

And since the divergence of the summation $\sum_{s=a}^{\infty} p_s$ implies that $p_s \neq 0$ it follows from Lemma 1 that

$$\Delta\omega_n + \omega_n^{\ 2} + \kappa\, p_n \leq 0$$

Thus for $n_0$ sufficiently large and $n \geq n_0$

$$\omega_n + \sum_{s=n_0}^{n} \omega_n^{\ 2} \leq \omega_{n_0} - \kappa \sum_{s=n_0}^{n} p_s < 0 ....................(5)$$

Let $h_n = \sum_{s=n_0}^{n} \omega_s^{\ 2}$ . Then it follows from (5) that $\Delta h_n \geq h_n^{\ 2}$ from which it follows that

$$n - n_o \leq \frac{1}{h_{n_0}} - \frac{1}{h_n} < \frac{1}{h_{n_0}}$$

For large $n$. Which contradiction to fact that $\omega_n \geq 0 \;\; for \;\; n \geq n_0$ .

If $r_n = 1, \{p_n\} \geq 0, g_n \to \infty \;\; as \;\; n \to \infty$ the all solutions equation (2) is oscillatory.

The following lemma is helpful is proving an oscillation theorem for (2) with non-constant $r$

If $f(y, w)$ has the $sign\, of \;\; y\, and \;\; w$ when they have the same sign, and $f(y, w)$ is non-decreasing in $y \;\; and \;\; w$ . For the oscillation Theorem presented here we make the following assumptions on $f \;\; and \;\; g$ :

(i) $g_n \to \infty \;\; as \;\; n \to \infty$

(ii) If $y \;\; and \;\; w$ are one sign, the $f(y, w)$ has that sign

(iii) $f(y, w)$ is bounded and zero.

Note that condition (ii) and (iii) is satisfied and $f$ is non decreasing in $y \;\; and \;\; w$ .

**Lemma 2**

$If \; \{p_n\} \geq 0, (p_n \neq 0), r_n \geq 0, \sum_{s=a}^{\infty} \frac{1}{r_s} = \infty$ Conditions (i)-(iii) hold and $\{u_n\}$ is a solutions of equation (1) that is positive, then $\Delta u_n \geq 0$ for all $n$ sufficiently large

**Proof**

If $\{u_n\} > 0 \;\; for$ large $n$ and lemma is false. Then there is a point $n_o$ larger than the last zero $u_n$ and larger than the last zero $u_{g(n)}$ such that $\Delta u_{n_0} < 0. \;\; for \;\; n \geq n_0$ $\Delta(r_n \Delta u_n) = -p_n f(u_n, u_{g(n)}) \leq 0$ and therefore $r_n \Delta u_n \leq r_{n_0} \Delta u_{n_0} < 0$ it follows dividing by $r_n$ and summing from $n_0 \;\; to \;\; n$ and $n \to \infty$ then $\{u_n\}$ is negative. This is contrary to the hypothesis. A similar argument treats the case of eventually positive or eventually negative If $\{u_n\} < 0 \;\; for$ large $n$

**Theorem 2**

$If \;\; p_n \geq 0, \; r_n \geq 0, \; \sum_{s=a}^{\infty} \frac{1}{r_s} = \infty, \; \sum_{s=a}^{\infty} p_s = \infty$ Conditions (i)-(iii) hold and $\{u_n\}$ is a solutions of equation (2) is oscillatory if interval $(a, \infty)$

**Proof:**

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Suppose there is a solutions $\{u_n\}$ on the interval $(a,\infty)$ and that If $\{u_n\} > 0$ *for* large $n$.

By lemma 2 $\Delta u_n \geq 0$ for all $n$ sufficiently large so that $\{u_n\}$ is non-decreasing; it then follows from conditions (i)-(iii) that there is positive constant $\ell$ such that $\ell \leq f(g(n), u_{g(n)})$ for all $n$ sufficiently large. Thus $\Delta(r_n\Delta(u_n) + p_n\ell \leq 0$ and for $n_o$ sufficiently and $n > n_o$ and Taking summation

$$r_n\Delta u_n - r_{n_0}\Delta u_{n_0} + \ell\sum_{s=n_0}^{n} p_s \leq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

But $\ell\sum_{s=n_0}^{n} p_s \to \infty \ as \ n \to \infty$ this contrary that to lemma. A similar argument left to reader if $u_n < 0$ *for* large $n$

### 3. Boundedness and Non-Oscillation

The proof of Lemma 2 suggests the following theorem

**Therom 3**

If (i)-(iii) hold $p_n \geq 0 \ and \{u_n\}$ is a non-oscillatory solution of (2) on an interval $(a,\infty)$, then there are non-negative constants $\ell_1, \ell_2$ such that $|u_n| \leq \ell_1 + \ell_2\sum_{s=n_0}^{n}\frac{1}{r_s}$. In particular, if $\sum_{s=n_0}^{n} p_s < \infty$ then all solutions existing on $(a,\infty)$ are oscillatory or bounded.

**Proof:**

If $\{u_n\}$ is non-oscillatory solution of (2) that is positive then $\Delta(r_n\Delta u_n) \leq 0,$ for all $n$ sufficentely large. Summing the $n_0$ *to* $n$

$$0 < u_n \leq u_{n_0} + r_{n_0}\Delta u_{n_0}\sum_{s=n_0}^{n}\frac{1}{r_s}$$

We take $\ell_1 = |u_{n_0}|, \ell_2 = r_{n_0}|\Delta u_{n_0}|$ if $\{u_n\}$ is negative for all large $n$, the process used above leads to the inequality

$$u_n \geq u_{n_0} + r_{n_0}\Delta u_{n_0}\sum_{s=n_0}^{n}\frac{1}{r_s}$$

In this case

$$0 < |u_n| = -u_n \leq -u_{n_0} - r_{n_0}\Delta u_{n_0}\sum_{s=n_0}^{n}\frac{1}{r_s}$$

This complete the proof

**Theorem 4**

Let $p_n \geq 0,$ suppose that $\{p_n\}$ is real sequence with $0 \leq \sigma(n) \leq m$ suppose that $\gamma \geq 1, \gamma$ is ratio of odd integers the signum fuctions :

$$\operatorname{sgn} u_n = 1 \text{ if } \{u_n\} > 0, \ \operatorname{sgn} u_n = -1 \text{ if } \{u_n\} < 0, \operatorname{sgn} 0 = 0$$

Then the condition

$$\sum_{s=a}^{\infty} sp_s = \infty \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Is necessary condition that all solutions of the equation

$$\Delta^2 u_n + p_n\left|(u_{(n-\sigma(n))})^{1-\gamma}\right|\operatorname{sgn} u_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

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Are oscillatory. For $\gamma = 1$, (7) is a sufficient condition that all bounded solutions of (8) are oscillatory.

**Proof:**

For $\gamma > 1$, , if $\{u_n\}$ is a bounded solution, and $\{u_n\} > 0$ for all $n$ sufficiently large, Then there ia point $c$ such that $\Delta^2 u_n \le 0, \Delta u_n \ge 0, u_n \ge 0,\ for\ n \ge c$ . then

$n \ge c, u_{n-m} \le u_{(n-\sigma(n))}$ so the multiplying $\dfrac{n}{u_{(n-\sigma(n))}}$ by (8)

and summing $c\ to\ n$

$$\sum_{s=c}^{n} \frac{s\Delta^2 u_s}{u_{s-m}} \le -\sum_{s=c}^{n} sp_s ..........................(9)$$

summing the fact that $\Delta u_{s-m} \ge \Delta u_s$ it is clear that (9) can be written as

$$\left[\frac{su_s}{u_{s-m}}\right]_c^n - \sum_{s=c}^{n} u_{s-m} + \sum_{s=c}^{n} \frac{s\Delta u_s \Delta u_{s-m}}{[u_{s-m}]^2} \le -\sum_{s=c}^{n} sp_s ........(10)$$

Since $\{u_n\}$ is bounded, the left side of (10) is bounded below while the right tends $-\infty$

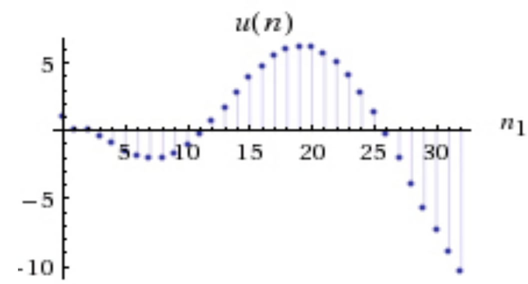
This contradiction completes the proof.

Example:1

Consider the difference equation

$$\Delta^2 u_n + \frac{n}{n^2+1} u_{n-1} = 0......................(11)$$

All condition to Theorem 1 to Theorem 4 satisfied and all solution of equation (11) is oscillatory.

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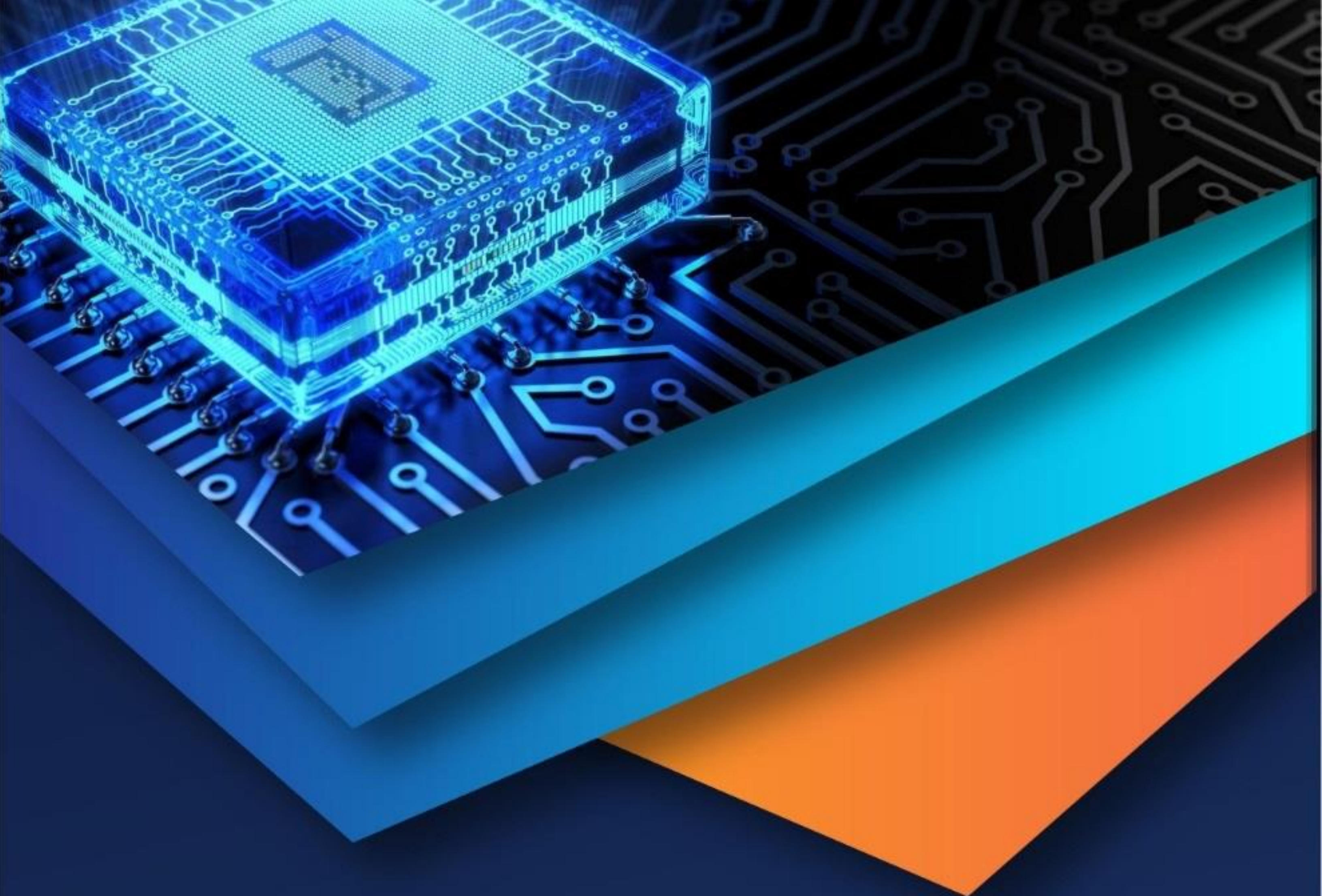


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