iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 6 **Issue:** V **Month of publication:** May 2018

DOI: http://doi.org/10.22214/ijraset.2018.5333

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 6 Issue V, May 2018- Available at www.ijraset.com

Colouring of Generalized Petersen Graph of Type-k

B. Stephen John[1], J. C. Jessy[2]
[1] Associate Professor and [2] Research Scholar Department of Mathematics, Annai Velankanni College, Tholayavattam, Tamilnadu. India-629157

Abstract: Let $G = \{V, E\}$ be a connected simple graph. A vertex colouring of a graph G is a function $f: V(G) \rightarrow C$, where C is a set of distinct colours. The vertex colouring problem is one of the fundamental problem on graphs which often appears in various scheduling problems like the file transfer problem on computer networks. In this paper we determine the vertex colouring of generalized petersen graph of type k.
Keywords: Cycle, Chromatic number, Vertex colouring, Generalized Petersen graph.

I. INTRODUCTION

The theory of graph colouring has existed for more than 150 years from its modest beginning of determining whether a geographic map can be coloured with four colours. The theory has become central in discrete mathematics with many contemporary generalization and application. In this paper, we are concerned with finite, connected, simple graph. Let $G = \{V(G), E(G)\}$ be a graph, if there is an edge e joining any two vertices u and v of G, we say that u and v are adjacent. A k-vertex colouring C of a graph G is an assignment of k-colours to the vertices of G.

A. Definition:1.1

A graph G is an ordered pair $(V(G), E(G))$ consisting of a non-empty set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$ of edges together with an incidence function ψ_G that associates with each edge of G is an unordered pair of vertices of G.

B. Definition: 1.2

A colouring of a simple connected graph G is colouring the vertices of G in such a way that no two adjacent vertices of G get the same colour. A graph is properly coloured if it is coloured with the minimum possible number of colours.

C. Definition: 1.3

The chromatic number of a graph G is the minimum number of colours required to colouring the vertices of G in properly and is denoted by $\chi(G)$.

D. Definition:1.4

The generalized petersen graph $GP(n,k)$ has vertices and edges of the form $V(GP(n,k)) = \{a_i, b_i / 0 \leq i \leq n-1\}$, $E(GP(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} / 0 \leq i \leq n-1\}$.

E. Definition: 1.5

Walk is an alternating sequence of vertices and edges starting and ending with vertices. A walk in which all the vertices are distinct is called a path. A path containing *n*-vertices is denoted by P_n. A closed path is called cycle. A cycle containing *n*-vertices is denoted by C_n, the length of a cycle is the number of edges occurring on it.

Let $G = GP(n,k)$ be the generalized petersen graph.

Let V(G) be the vertex set of $G = GP(n,k)$. It can be partitioned into two sets V_1 and V_2 such that

i. $V_1(G) = \{v_i / i = 1,2, \dots, n\}$
ii. $V_2(G) = \{u_i / i = 1,2, \dots, n\}$

Clearly, the vertices of $V_{i;\{i=1,2\}}$ satisfies the condition $V_1(G) \cap V_2(G) = \emptyset$. Also, each vertex $v_i \epsilon V_1$ is adjacent to the corresponding $u_i \epsilon V_2$.

The elements of $V_1(G)$ form a cycle of length n. Let it be C_1.

1) *Type: I:* If $gcd(n, k) = k$ and $\frac{n}{k}$ is even, the vertex set of V_2 contains k-disjoint cycles of length n/k. Let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_k}\}$.

For example, $GP(12,3)$ is represented in figure:2.1

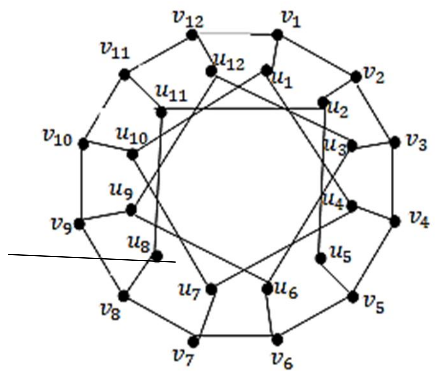

Figure:2.1

Clearly, the cycles C_1 and $C_{2_i\,;\,(i=1,2,\dots,k)}$ are of even in length. By lemma:1, We need 2-colours to colour all the vertices of V_1. Let it be c_1 and c_2. Suppose the colour c_1 is given to the vertex v_1. Since, each v_i is adjacent with the corresponding u_i, we can't assign the colour c_1 to the vertex u_1. Hence, we can assign the cour c_2 to the vertex u_1. Since the cycles $C_{2_i\,;\,(i=1,2,\dots,k)}$ are even in length, repeat the above process for each u_i and from these above processes we can conclude that, in this type we need 2-colours to colour all the vertices of G. Therefore, $\chi(G) = 2$.

2) *Type: II* If $gcd(n, k) = 1$, the vertex set of V_2 contains a cycle of length n. Let it be C_2 For example, $GP(8,3)$ is represented in figure:2.2

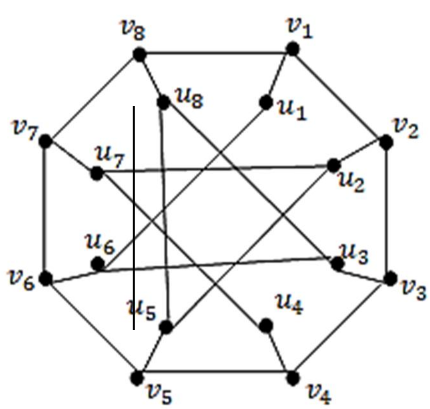

Figure:2.2

Since, the cycle C_1 is of even in length. By lemma:1, We need 2-colours to colour all the vertices of C_1. Let it be c_1 and c_2. Now, fix the colour c_1 to the vertex v_1. BSince, each v_i is adjacent with the corresponding u_i, the vertices of C_2 are coloured according to their adjacency with the vertices of the cycle C_1. Hence, we need 2-colours to colour all the vertices of G. Therefore, $\chi(G) = 2$.

3) *Theorem:2.2* Let G be the generalized petersen graph of type k, that is $G = GP(n, k)$, for all *n* and k are odd, then $\chi(G) = 3$.
4) *Proof:* Let $G = GP(n, k)$ be the generalized petersen graph. Let V(G) be the vertex set of $G = GP(n, k)$. It can be partitioned into two sets V_1 and V_2 such that
   i. $V_1(G) = \{v_i / i = 1,2, \dots, n\}$
   ii. $V_2(G) = \{u_i / i = 1,2, \dots, n\}$

Clearly, the vertices of $V_{i;\{i=1,2\}}$ satisfies the condition $V_1(G) \cap V_2(G) = \emptyset$. Also, each vertex $v_i \in V_1$ is adjacent to the corresponding $u_i \in V_2$. The elements of $V_1(G)$ form a cycle of length n. Let it be C_1.

5) *Type: I:* If $gcd(n,k) = k$ and n/k is odd.

The vertex set of V_2 contains k-disjoint cycles of length n/k. Let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_k}\}$. For example, $GP(9,3)$ is represented in figure:2.3

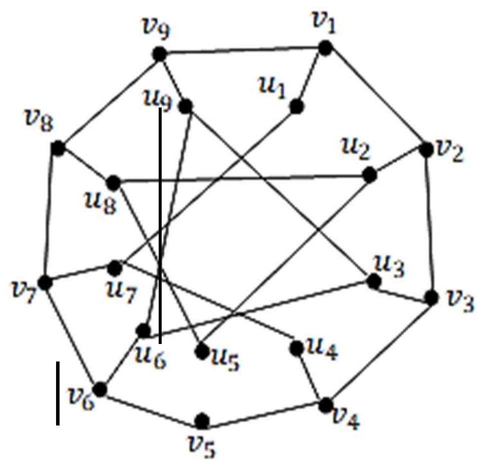

Figure:2.3

Clearly, the cycles C_1 and $C_{2_i\,;\,(i=1,2,\dots,k)}$ are of odd in length. Fix the vertex v_1 and assign the colour c_1 to the vertex v_1. Since the vertex v_2 is adjacent to the vertex v_1, we need a new colour c_2 to colour v_2. Also, the vertex v_3 is independent with v_1 and adjacent with v_2, so we can give the colour c_1 to $v_3 \in V_1$. Repeat the above process for each $v_{i\,;\,(i=1,2,\dots,n-1)}$. Since, the cycle C_1 is of odd in length. The vertex v_n is adjacent to both the colours c_1 and c_2, so we need another new colour c_3 to colour the vertex v_n. Hence, we need 3-colours to colour all the vertices of the cycle C_1. Now, to colour the vertices of V_2, assign the colours c_1 and c_2 to the vertices $u_{j\,;(j=1,2,\dots,k-1)} \in V_2$ according to its adjacency with the corresponding $v_j \in V_1$ Repeat the above process for each $u_{j+k} \in V_2$ by assigning the colours c_1 and c_2. Also, the vertices $u_{n-k+j} \in V_2$ are independent with the vertices which are coloured by the colour c_3 and adjacent with the remaining two colours c_1 and c_2. So, we can assign the colour c_3 to the (u_{n-k+j})-vertices. Now, we can colour the remaining (u_{jk})-vertices of the cycle C_{2_k} by fixing the colour c_1 to the vertex u_n and assigning the colours c_1, c_2 and c_3 to all other (u_{jk})-vertices according to its adjacency with other vertices. Hence, we need 3-colours to colour all the vertices of G. Therefore, $\chi(G) = 3$.

6) *Type: II* If $gcd(n,k) = 1$. the vertex set V_2 form a cycle C_2 of odd length. Therefore, both the cycles $C_{2_i\,;\,(i=1,2)}$ are of odd length For example, $GP(7,3)$ is represented in figure:2.4

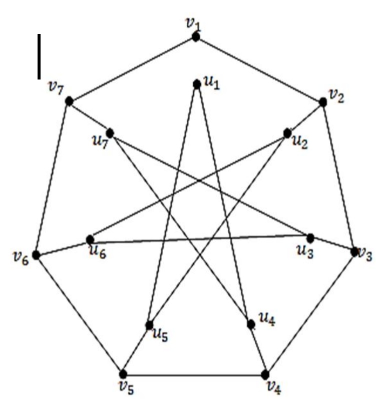

Figure:2.4

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 6 Issue V, May 2018- Available at www.ijraset.com

Fix the vertex v_1 and assign the colour c_1 to the vertex v_1.
into two sets V_1 and V_2 such that$V_1(G) = \{v_i / i = 1,2, \dots, n\}$
$V_2(G) = \{u_i / i = 1,2, \dots, n\}$
Clearly, the vertices of $V_{i;\{i=1,2\}}$ satisfies the condition $V_1(G) \cap V_2(G) = \emptyset$. Each vertex v_i is adjacent to the corresponding u_i.
The elements of $V_1(G)$ form a cycle of length n. Let it be C_1.
7) Type: I If $gcd(n,k) = 1$For example, $GP(9,2)$ is represented in figure:2.5

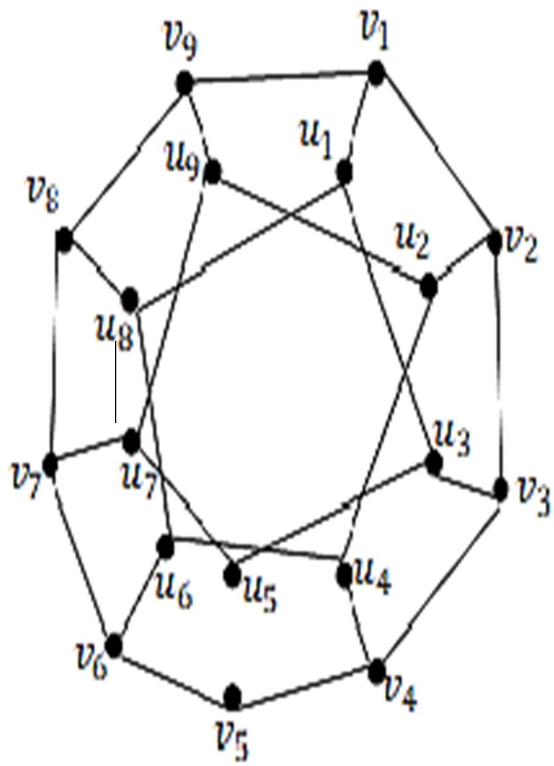

Figure:2.5

Clearly, the vertex set V_2 form a cycle C_2 of odd length.
Hence, both the cycles $C_{2_i\,;\,(i=1,2)}$ are of odd length.
Fix the ertex v_1 and assign the colour c_1 to the vertex v_1.
Since the vertex v_2 is adjacent to the vertex v_1, we need a new colour c_2 to colour v_2.
Also, the vertex v_3 is independent with v_1 and adjacent with v_2, so we can give the colour c_1 to $v_3 \epsilon V_1$.
Repeat the above process for each $v_{i\ ;\ (i=1,2,\dots,n-1)}$.
Since, the cycle C_1 is of odd in length. The vertex v_n is adjacent to both the colours c_1 and c_2, so we need another new colour c_3 to colour the vertex v_n.
Hence, we need 3-colours to colour all the vertices of the cycle C_1.
Now, we can colour the vertices of the vertex set V_2 by fixing the colour c_2 to the vertex u_n and colour c_1 to the vertex u_k.
The independent vertices $u_{k+1}, u_{k+2}, u_{n-1}$ and u_{n-2} are adjacent to the vertices which are coloured by the colours c_1 and c_2 and independent to the colour c_3. So, we can give the colour c_3 to these four vertices. The remaining u_j-vertices are coloured by the colours c_1 and c_2 according to its adjacency withal the other vertices.
Hence, in this type we need 3-colours to colour all the vertices of G.Therefore, $\chi(G) = 3$.
If $gcd(n,k) = t$ and n/t is odd. The vertex set of V_2 contains t-disjoint cycles of length n/t. Let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$.
For example, $GP(5,6)$ is represented in figure:2.6

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 6 Issue V, May 2018- Available at www.ijraset.com

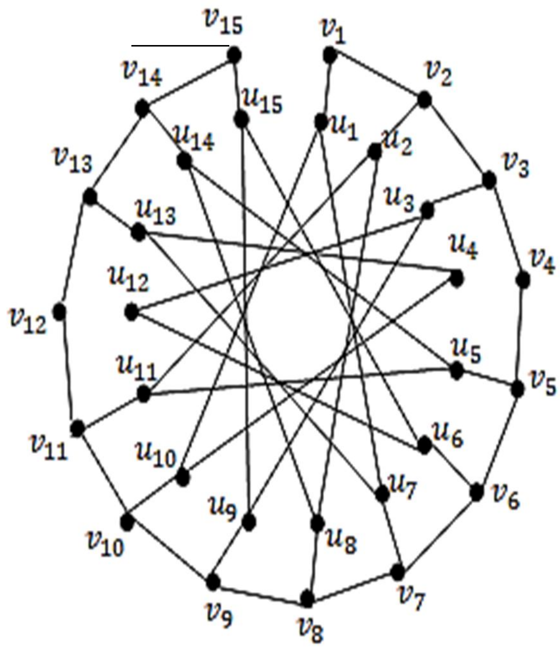

Figure:2.6

Clearly, the cycles C_1 and $C_{2_i\,;\,(i=1,2,\dots,t)}$ are of odd in length.
Fix the vertex v_1 and assign the colour c_1 to the vertex v_1.
Since the vertex v_2 is adjacent to the vertex v_1, we need a new colour c_2 to colour v_2.
Also, the vertex v_3 is independent with v_1 and adjacent with v_2, so we can give the colour c_1 to $v_3 \in V_1$.
Repeat the above process for each $v_{i\,;\,(i=1,2,\dots,n-1)}$. Since, the cycle C_1 is of odd in length. The vertex v_n is adjacent to both the colours c_1 and c_2, so we need another new colour c_3 to colour the vertex v_n.
Hence, we need 3-colours to colour all the vertices of the cycle C_1.
Now, t colour the vertices of V_2, assign the colours c_1 and c_2 to the vertices $u_{j\,;(j=1,2,\dots,k-1)} \in V_2$ according to its adjacency with the corresponding $v_j \in V_1$.
Repeat the above process for each $u_{j+k} \in V_2$ by assigning the colours c_1 and c_2. Also, the vertices $u_{n-k+j} \in V_2$ are independent with the vertices which are coloured by the colour c_3 and adjacent with the remaining two colours c_1 and c_2. So, we can assign the colour c_3 to the (u_{n-k+j})-vertices.
Now, we can colour the remaining (u_{jk})-vertices of the cycle C_{2_k} by fixing the colour c_1 to the vertex u_n and assigning the colours c_1, c_2 and c_3 to all other (u_{jk})-vertices according to its adjacency with other vertices.
Hence, we need 3-colours to colour all the vertices of G.
Therefore, $\chi(G) = 3$.

8) *Theorem:2.4* Let G be the generalized petersen graph of type k, that is $G = GP(n,k)$, for all n and k are even, then $\chi(G) = 3$.
9) *Proof:* Let $G = GP(n,k)$ be the generalized petersen graph.
(G) be the vertex set of $G = GP(n,k)$. It can be partitioned into two sets V_1 and V_2 such that
i. $V_1(G) = \{v_i / i = 1,2,\dots,n\}$
ii. $V_2(G) = \{u_i / i = 1,2,\dots,n\}$
From (i) and (ii), the vertices of $V_{i;\{i=1,2\}}$ satisfies the condition $V_1(G) \cap V_2(G) = \emptyset$.
The elements of $V_1(G)$ form a cycle of length n. Let it be C_1.

10) *Type: I* If $gcd(n,k) = k$. For example, $GP(8,2)$ is represented in figure:2.7

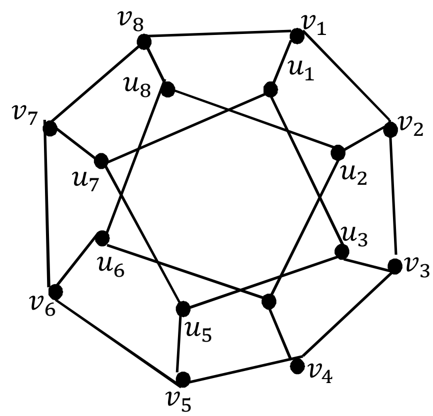

Figure:2.7

Since the cycle C_1 is of even length.
By lema:1,
We need 2-colours to colour all the vertices of C_1. Let it be c_1 and c_2.
The vertices of V_2 split into k-cycles. Let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_k}\}$. Each cycle $C_{2_i\,;\,(i=1,2,\dots,k)}$ is of even in length.
The vertex $\{v_{i\,;(i=1,2,\dots,k)}\}$ is adjacent to the vertex $\{u_{i\,;\,(i=1,2,\dots,k)}\}$ which is coloured by the colour c_1.Therefore, we can assign the colour c_2 to the vertex u_i.
But the vertices $\{u_{i+k\,;\,(i=1,2,\dots,k)}\}$ and $\{u_{n-k+i\,;\,(i=1,2,\dots,k)}\}$ are adjacent with the vertices which are coloured by the colours c_1 and c_2.
Hence, we need a new colour c_3 to colours these two disjoint vertices.
Hence, we need 3-colours to colour all the vertices of this type.
Therefore, $\chi(G) = 3$.

11) Type: II If $gcd(n, k) = t$ and n/t is odd. For example, $GP(10,4)$ is represented in figure:2.8

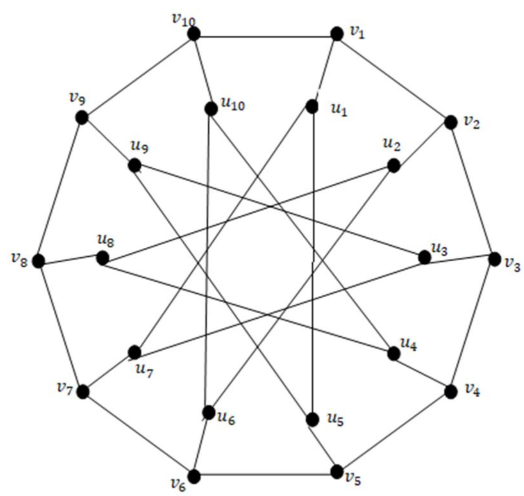

Figure:2.8

Since, the cycle C_1 is of even in length.
By lemma:1,
We need 2-colours to colour all the vertices of C_1. Let it be c_1 and c_2.
The vertices of V_2 split into t-cycles. Let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$. Each cycle $C_{2_i\,;\,(i=1,2,\dots,t)}$ are of odd in length.
By lemma:1,

We need 3-colours to colour all the vertices $C_{2_i\,;\,(i=1,2,\dots,t)}$ according to there adjacency with the vertices of C_1.

Therefore, we need 3-colours to colour all the vertices of this type.
Hence, $\chi(G) = 3$.

*12) Type: III*If $gcd(n,k) = t$ and n/t is even.For example, $GP(16,6)$ is represented in figure:2.9

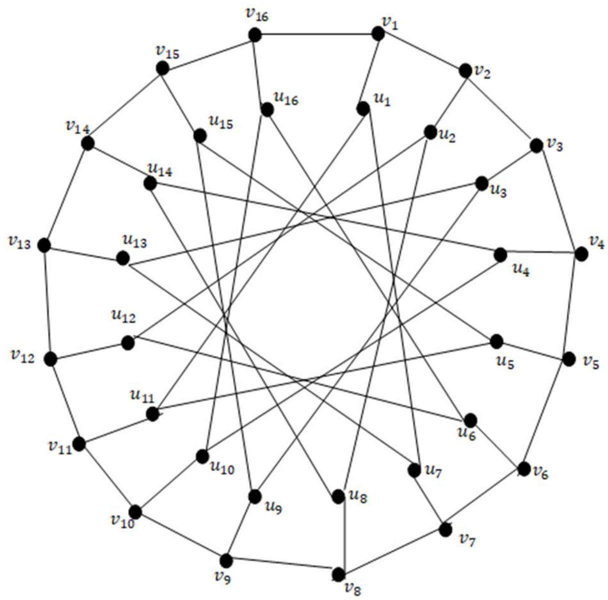

Figure:2.9

Since, the cycle C_1 is of even in length.
By lemma:1,
We need 2-colours to colour all the vertices of C_1. Let it be c_1 and c_2.
The vertices of V_2 split into k-cycles. Let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$. Each cycle $C_{2_i\,;\,(i=1,2,\dots,t)}$ is of even in length.
The vertex $v_{i\,;(i=1,2,\dots,t)}$ is adjacent to the vertex $u_{i\,;\,(i=1,2,\dots,t)}$ which is coloured by the colour c_1. Thererfore, we can assign the colour c_2 to the vertex u_i.
But the vertices $u_{i+k\,;\,(i=1,2,\dots,t)}$ and $u_{n-k+i\,;\,(i=1,2,\dots,t)}$ are adjacent with the vertices which are coloured by the colours c_1 and c_2. Hence, we need a new colour c_3 to colours these two disjoint vertices.
Hence, we need 3-colours to colour all the vertices of this type.
Therefore, $\chi(G) = 3$. Hence proved.

REFERENCES

[1] Robento.W.Frucht, Jack.E.Graver and Mark.E.Watkins, “The groups of the generalized petersen graph”, Proc. Cambridge Philas. Sec.,70(1971), 211-218.
[2] Halton.D.A and Sheehan.J, “Generalized Petersen and Permutation Graphs”, Cambridge, England; Cambridge University Press, pp-45 and 315-317, 1998.
[3] Ponstein.J.(1969). A New Proof of Brooks’, “Chromatic Number Theorem for Graphs”, Journal of Combinatorial theory, 7, 255-257.
[4] Dantas.S, de Figueiredo.C.M.H, Maxzuoccoloc.G, Preissmann.M, dos Santos.V.F and Sasaki.D(2016) “On the Total Colouring of Generalized Petersen Graphs”, Discrete Mathematics, 339, 1471-1475.