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Observation on the Negative Pell Equation $y^2 = 40x^2 - 36$

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Abstract: The binary quadratic equation represented by the negative pellian $y^2 = 40x^2 - 36$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas. Also, the relations between the solutions and special figurate numbers are exhibited.
Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integer solutions and Figurate numbers.

I. INTRODUCTION

The subject of Diophantine equation is one of the areas in Number Theory that has attracted many Mathematicians since antiquity and it has a long history. Obviously, the Diophantine equation are rich in variety [1-3]. In particular, the binary quadratic diophantine equation of the form $y^2 = Dx^2 - N\ (N > 0,\ D > 0\ \text{and square free})$ is referred as the negative form of the pell equation (or) related pell equation. It is worth to observe that the negative pell equation is not always solvable. For example, the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solutions whereas $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [4-10] for a few negative pell equations with integer solutions.

In this communication, the negative pell equation given by $y^2 = 40x^2 - 36$ is considered and analysed for its integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas. Also, the relations between the solutions and special figurate numbers are exhibited.

II. NOTATIONS

1) $t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$ - Polygonal number of rank n with size m

2) $p_n^m = \frac{1}{6}n(n+1)[(m-2)n + (5-m)]$ - Pyramidal number of rank n with size m

III. METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 40x^2 - 36 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1,\ y_0 = 2$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 40x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{4\sqrt{10}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(19 + 3\sqrt{40}\right)^{n+1} + \left(19 - 3\sqrt{40}\right)^{n+1}$$

$$g_n = \left(19+3\sqrt{40}\right)^{n+1} - \left(19-3\sqrt{40}\right)^{n+1} \ , \ n=-1,0,1,2......$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{1}{2\sqrt{10}} g_n$$

$$y_{n+1} = f_n + \sqrt{10} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 38x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 38y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table: 1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	1	2
0	25	158
1	949	6002
2	36037	227918
3	1368457	8654882
4	51965329	328657598

From the above table, we observe some interesting relations among the solutions which are presented below:

A. x_{n+1} *is always odd,* y_{n+1} *is always even.*

B. Relations Among the Solutions

1) $x_{n+2} - 19x_{n+1} - 3y_{n+1} = 0$
2) $27y_{n+2} - 171x_{n+2} + 9x_{n+1} = 0$
3) $3y_{n+3} - 721x_{n+2} + 19x_{n+1} = 0$
4) $x_{n+3} - 38x_{n+2} + x_{n+1} = 0$
5) $19x_{n+3} - 721x_{n+2} - 3y_{n+1} = 0$
6) $x_{n+3} - 19x_{n+2} - 3y_{n+2} = 0$
7) $19x_{n+3} - x_{n+2} - 3y_{n+3} = 0$
8) $x_{n+3} - 114y_{n+1} - 721x_{n+1} = 0$
9) $x_{n+3} - 6y_{n+2} - x_{n+1} = 0$
10) $721x_{n+3} - x_{n+1} - 114y_{n+3} = 0$
11) $y_{n+2} - 120x_{n+1} - 19y_{n+1} = 0$
12) $y_{n+3} - 4560x_{n+1} - 721y_{n+1} = 0$
13) $19y_{n+3} - 721y_{n+2} - 120x_{n+1} = 0$
14) $120x_{n+2} - 19y_{n+2} + y_{n+1} = 0$
15) $y_{n+3} - 240x_{n+2} - y_{n+1} = 0$
16) $y_{n+3} - 120x_{n+2} - 19y_{n+2} = 0$
17) $120x_{n+3} - 721y_{n+2} + 19y_{n+1} = 0$
18) $721y_{n+3} - 4560x_{n+3} - y_{n+1} = 0$
19) $120x_{n+3} - 19y_{n+3} + y_{n+2} = 0$
20) $y_{n+3} - 38y_{n+2} + y_{n+1} = 0$

C. Each Of The Following Expressions Represents A Nasty Number

1) $\frac{2}{9}(79x_{2n+2} - x_{2n+3} + 54)$

2) $\frac{2}{9}(3001x_{2n+3} - 79x_{2n+4} + 54)$

3) $\frac{1}{171}(3001x_{2n+2} - x_{2n+4} + 2052)$

4) $\frac{2}{3}(20x_{2n+2} - y_{2n+2} + 18)$

5) $\frac{2}{57}(500x_{2n+2} - y_{2n+3} + 342)$

6) $\frac{2}{2163}(18980x_{2n+2} - y_{2n+4} + 12978)$

7) $\frac{2}{57}(20x_{2n+3} - 79y_{2n+2} + 342)$

8) $\frac{2}{3}(500x_{2n+3} - 79y_{2n+3} + 18)$

9) $\frac{2}{57}(18980x_{2n+3} - 79y_{2n+4} + 342)$

10) $\frac{2}{2163}(20x_{2n+4} - 3001y_{2n+2} + 12978)$

11) $\frac{2}{57}(500x_{2n+4} - 3001y_{2n+3} + 342)$

12) $\frac{2}{3}(18980x_{2n+4} - 3001y_{2n+4} + 18)$

13) $\frac{1}{9}(y_{2n+3} - 25y_{2n+2} + 108)$

14) $\frac{1}{342}(y_{2n+4} - 949y_{2n+2} + 4104)$

15) $\frac{1}{9}(25y_{2n+4} - 949y_{2n+3} + 108)$

D. Each Of The Following Expressions Represents A Cubical Integer

1) $\frac{1}{27}[79x_{3n+3} - x_{3n+4} + 237x_{n+1} - 3x_{n+2}]$

2) $\frac{1}{27}[3001x_{3n+4} - 79x_{3n+5} + 9003x_{n+2} - 237x_{n+3}]$

3) $\frac{1}{1026}[3001x_{3n+3} - x_{3n+5} + 9003x_{n+1} - 3x_{n+3}]$

4) $\frac{1}{9}[20x_{3n+3} - y_{3n+3} + 60x_{n+1} - 3y_{n+1}]$

5) $\frac{1}{171}[500x_{3n+3} - y_{3n+4} + 1500x_{n+1} - 3y_{n+2}]$

6) $\frac{1}{6489}[18980x_{3n+3} - y_{3n+5} + 56940x_{n+1} - 3y_{n+3}]$

7) $\frac{1}{171}\left[20x_{3n+4}-79y_{3n+3}+60x_{n+2}-237y_{n+1}\right]$

8) $\frac{1}{9}\left[500x_{3n+4}-79y_{3n+4}+1500x_{n+2}-237y_{n+2}\right]$

9) $\frac{1}{171}\left[18980x_{3n+4}-79y_{3n+5}+56940x_{n+2}-237y_{n+3}\right]$

10) $\frac{1}{6489}\left[20x_{3n+5}-3001y_{3n+3}+60x_{n+3}-9003y_{n+1}\right]$

11) $\frac{1}{171}\left[500x_{3n+5}-3001y_{3n+4}+1500x_{n+3}-9003y_{n+2}\right]$

12) $\frac{1}{9}\left[18980x_{3n+5}-3001y_{3n+5}+56940x_{n+3}-9003y_{n+3}\right]$

13) $\frac{1}{54}\left[y_{3n+4}-25y_{3n+3}+3y_{n+2}-75y_{n+1}\right]$

14) $\frac{1}{2052}\left[y_{3n+5}-949y_{3n+3}+3y_{n+3}-2847y_{n+1}\right]$

15) $\frac{1}{54}\left[25y_{3n+5}-949y_{3n+4}+75y_{n+3}-2847y_{n+2}\right]$

E. Each Of The Following Expressions Represents A Bi-Quadratic Integer

1) $\frac{1}{27}\left[79x_{4n+4}-x_{4n+5}+316x_{2n+2}-4x_{2n+3}+162\right]$

2) $\frac{1}{27}\left[3001x_{4n+5}-79x_{4n+6}+12004x_{2n+3}-316x_{2n+4}+162\right]$

3) $\frac{1}{1026}\left[3001x_{4n+4}-x_{4n+6}+12004x_{2n+2}-4x_{2n+4}+6156\right]$

4) $\frac{1}{9}\left[20x_{4n+4}-y_{4n+4}+80x_{2n+2}-4y_{2n+2}+54\right]$

5) $\frac{1}{171}\left[500x_{4n+4}-y_{4n+5}+2000x_{2n+2}-4y_{2n+3}+1026\right]$

6) $\frac{1}{6489}\left[18980x_{4n+4}-y_{4n+6}+75920x_{2n+2}-4y_{2n+4}+38934\right]$

7) $\frac{1}{171}\left[20x_{4n+5}-79y_{4n+4}+80x_{2n+3}-316y_{2n+2}+1026\right]$

8) $\frac{1}{9}\left[500x_{4n+5}-79y_{4n+5}+2000x_{n+2}-316y_{n+2}+54\right]$

9) $\frac{1}{171}\left[18980x_{4n+5}-79y_{4n+6}+75920x_{2n+3}-316y_{2n+4}+1026\right]$

10) $\frac{1}{6489}\left[20x_{4n+6}-3001y_{4n+4}+80x_{2n+4}-12004y_{2n+2}+38934\right]$

11) $\frac{1}{171}\left[500x_{4n+6}-3001y_{4n+5}+2000x_{2n+4}-12004y_{2n+3}+1026\right]$

12) $\frac{1}{9}\left[18980x_{4n+6}-3001y_{4n+6}+75920x_{2n+4}-12004y_{2n+4}+54\right]$

13) $\frac{1}{54}\left[y_{4n+5}-25y_{4n+4}+4y_{2n+3}-100y_{2n+2}+324\right]$

14) $\frac{1}{2052}\left[y_{4n+6}-949y_{4n+4}+4y_{2n+4}-3796y_{2n+2}+12312\right]$

15) $\frac{1}{54}\left[25y_{4n+6}-949y_{4n+5}+100y_{2n+4}-3796y_{2n+3}+324\right]$

F. Each Of The Following Expressions Represents A Quintic Integer

1) $\frac{1}{27}\left[79x_{5n+5}-x_{5n+6}+395x_{3n+3}-5x_{3n+4}+790x_{n+1}-10x_{n+2}\right]$

2) $\frac{1}{27}\left[3001x_{5n+6}-79x_{5n+7}+15005x_{3n+4}-395x_{3n+5}+30010x_{n+2}-790x_{n+3}\right]$

3) $\frac{1}{1026}\left[3001x_{5n+5}-x_{5n+7}+15005x_{3n+3}-5x_{3n+5}+30010x_{n+1}-10x_{n+3}\right]$

4) $\frac{1}{9}\left[20x_{5n+5}-y_{5n+5}+100x_{3n+3}-5y_{3n+3}+200x_{n+1}-10y_{n+1}\right]$

5) $\frac{1}{171}\left[500x_{5n+5}-y_{5n+6}+2500x_{3n+3}-5y_{3n+4}+5000x_{n+1}-10y_{n+2}\right]$

6) $\frac{1}{6489}\left[18980x_{5n+5}-y_{5n+7}+94900x_{3n+3}-5y_{3n+5}+189800x_{n+1}-10y_{n+3}\right]$

7) $\frac{1}{171}\left[20x_{5n+6}-79y_{5n+5}+100x_{3n+4}-395y_{3n+3}+200x_{n+2}-790y_{n+1}\right]$

8) $\frac{1}{9}\left[500x_{5n+6}-79y_{5n+6}+2500x_{3n+4}-395y_{3n+4}+5000x_{n+2}-790y_{n+2}\right]$

9) $\frac{1}{171}\left[18980x_{5n+6}-79y_{5n+7}+94900x_{3n+4}-395y_{3n+5}+189800x_{n+2}-790y_{n+3}\right]$

10) $\frac{1}{6489}\left[20x_{5n+7}-3001y_{5n+5}+100x_{3n+5}-15005y_{3n+3}+200x_{n+3}-30010y_{n+1}\right]$

11) $\frac{1}{171}\left[500x_{5n+7}-3001y_{5n+6}+2500x_{3n+5}-15005y_{3n+4}+5000x_{n+3}-30010y_{n+2}\right]$

12) $\frac{1}{9}\left[18980x_{5n+7}-3001y_{5n+7}+94900x_{3n+5}-15005y_{3n+5}+189800x_{n+3}-30010y_{n+3}\right]$

13) $\frac{1}{54}\left[y_{5n+6}-25y_{5n+5}+5y_{3n+4}-125y_{3n+3}+10y_{n+2}-250y_{n+1}\right]$

14) $\frac{1}{2052}\left[y_{5n+7}-949y_{5n+5}+5y_{3n+5}-4745y_{3n+3}+10y_{n+3}-9490y_{n+1}\right]$

15) $\frac{1}{54}\left[25y_{5n+7}-949y_{5n+6}+125y_{3n+5}-4745y_{3n+4}+250y_{n+3}-9490y_{n+2}\right]$

G. Remarkable Observations

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbola	(X, Y)
1	$X^2 - 10Y^2 = 2916$	$(79x_{n+1} - x_{n+2}, x_{n+2} - 25x_{n+1})$
2	$X^2 - 10Y^2 = 2916$	$(3001x_{n+2} - 79x_{n+3}, 25x_{n+3} - 949x_{n+2})$
3	$X^2 - 10Y^2 = 4210704$	$(3001x_{n+1} - x_{n+3}, x_{n+3} - 949x_{n+1})$
4	$X^2 - 10Y^2 = 324$	$(20x_{n+1} - y_{n+1}, y_{n+1} - 2x_{n+1})$
5	$X^2 - 10Y^2 = 116964$	$(500x_{n+1} - y_{n+2}, y_{n+2} - 158x_{n+1})$
6	$X^2 - 10Y^2 = 168428484$	$(18980x_{n+1} - y_{n+3}, y_{n+3} - 6002x_{n+1})$
7	$X^2 - 10Y^2 = 116964$	$(20x_{n+2} - 79y_{n+1}, 25y_{n+1} - 2x_{n+2})$
8	$X^2 - 10Y^2 = 324$	$(500x_{n+2} - 79y_{n+2}, 25y_{n+2} - 158x_{n+2})$
9	$X^2 - 10Y^2 = 116964$	$(18980x_{n+2} - 79y_{n+3}, 25y_{n+3} - 6002x_{n+2})$
10	$X^2 - 10Y^2 = 168428484$	$(20x_{n+3} - 3001y_{n+1}, 949y_{n+1} - 2x_{n+3})$
11	$X^2 - 10Y^2 = 116964$	$(500x_{n+3} - 3001y_{n+2}, 949y_{n+2} - 158x_{n+3})$
12	$X^2 - 10Y^2 = 324$	$(18980x_{n+3} - 3001y_{n+3}, 949y_{n+3} - 6002x_{n+3})$
13	$10X^2 - Y^2 = 116640$	$(y_{n+2} - 25y_{n+1}, 79y_{n+1} - y_{n+2})$
14	$10X^2 - Y^2 = 168428160$	$(y_{n+3} - 949y_{n+1}, 3001y_{n+1} - y_{n+3})$
15	$10X^2 - Y^2 = 116640$	$(25y_{n+3} - 949y_{n+2}, 3001y_{n+2} - 79y_{n+3})$

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2) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabola	(X,Y)
1	$27X-10Y^2=2916$	$(79x_{2n+2}-x_{2n+3}+54,\ x_{n+2}-25x_{n+1})$
2	$27X-10Y^2=2916$	$(3001x_{2n+3}-79x_{2n+4}+54,\ 25x_{n+3}-949x_{n+2})$
3	$1026X-10Y^2=4210704$	$(3001x_{2n+2}-x_{2n+4}+2052,\ x_{n+3}-949x_{n+1})$
4	$9X-10Y^2=324$	$(20x_{2n+2}-y_{2n+2}+18,\ y_{n+1}-2x_{n+1})$
5	$171X-10Y^2=116964$	$(500x_{2n+2}-y_{2n+3}+342,\ y_{n+2}-158x_{n+1})$
6	$6489X-10Y^2=168428484$	$(18980x_{2n+2}-y_{2n+4}+12978,\ y_{n+3}-6002x_{n+1})$
7	$171X-10Y^2=116964$	$(20x_{2n+3}-79y_{2n+2}+342,\ 25y_{n+1}-2x_{n+2})$
8	$9X-10Y^2=324$	$(500x_{2n+3}-79y_{2n+3}+18,\ 25y_{n+2}-158x_{n+2})$
9	$171X-10Y^2=116964$	$(18980x_{2n+3}-79y_{2n+4}+342,\ 25y_{n+3}-6002x_{n+2})$
10	$6489X-10Y^2=168428484$	$(20x_{2n+4}-3001y_{2n+2}+12978,\ 949y_{n+1}-2x_{n+3})$
11	$171X-10Y^2=116964$	$(500x_{2n+4}-3001y_{2n+3}+342,\ 949y_{n+2}-158x_{n+3})$
12	$9X-10Y^2=324$	$(18980x_{2n+4}-3001y_{2n+4}+18,\ 949y_{n+3}-6002x_{n+3})$
13	$540X-Y^2=116640$	$(y_{2n+3}-25y_{2n+2}+108,\ 79y_{n+1}-y_{n+2})$
14	$20520X-Y^2=168428160$	$(y_{2n+4}-949y_{2n+2}+4104,\ 3001y_{n+1}-y_{n+3})$
15	$540X-Y^2=116640$	$(25y_{2n+4}-949y_{2n+3}+108,\ 3001y_{n+2}-79y_{n+3})$

3) Relations between the solutions and special figurate numbers

a) $9(p_y^3 * t_{3,x})^2 = 40(p_x^5 * t_{3,y+1})^2 - 36(t_{3,x} * t_{3,y+1})^2$

b) $(p_y^5 * t_{3,x+1})^2 = 360(p_x^3 * t_{3,y})^2 - 36(t_{3,y} * t_{3,x+1})^2$

c) $36(p_{y-1}^4 * t_{3,x})^2 = 40(p_x^5 * t_{3,2y-2})^2 - 36(t_{3,x} * t_{3,2y-2})^2$

d) $(p_y^5 * t_{3,2x-2})^2 = 1440(p_{x-1}^4 * t_{3,y})^2 - 36(t_{3,y} * t_{3,2x-2})^2$

e) $(p_{y-1}^4 * t_{3,x+1})^2 = 10(p_x^3 * t_{3,2y-2})^2 - (t_{3,x+1} * t_{3,2y-2})^2$

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f) $\left(p_y^3 * t_{3,2x-2}\right)^2 = 160\left(p_{x-1}^4 * t_{3,y+1}\right)^2 - 4\left(t_{3,y+1} * t_{3,2x-2}\right)^2$

g) Let $\{n_{s+1}\}$ and $\{m_{s+1}\}$ be sequences of positive integers defined by

$$\{n_{s+1}\} = \frac{1}{2}\left(x_{s+1} - 1\right), \qquad \{m_{s+1}\} = \frac{1}{2}\left(y_{s+1}\right)$$

Note that,

$480t_{3,n_{s+1}} + 6$ is a nasty number

$$160t_{3,n_{s+1}} - t_{6,m_{s+1}} = m_{s+1} - 2$$

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