International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue I, Jan 2019- Available at www.ijraset.com

On the Positive Pell Equation $y^2 = 35x^2 + 14$

K. Meena[1], M. A. Gopalan[2], T. Priya Lakshmi [3]
[1]Former VC, Bharathidasan University, Trichy-620 024.Tamilnadu, India
[2]Professor, Department of Mathematics, SIGC, Trichy, India.
[3]M.Phil Scholar, Department of Mathematics, SIGC, Trichy, India

***Abstract:** The binary quadratic equation represented by the positive Pellian $y^2 = 35x^2 + 14$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.*

***Keywords:** Binary quadratic, hyperbola, integral solutions, parabola, pell equation. 2010 mathematics subject classification: 11D09*

I. INTRODUCTION

A binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-12]. In this communication, yet another interesting hyperbola given by $y^2 = 35x^2 + 14$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained.

A. Method of Analysis

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 35x^2 + 14 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, \; y_0 = 7$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 35x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{35}} g_n \, , \; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(6 + \sqrt{35}\right)^{n+1} + \left(6 - \sqrt{35}\right)^{n+1}$$

$$g_n = \left(6 + \sqrt{35}\right)^{n+1} - \left(6 - \sqrt{35}\right)^{n+1} \quad , \; n = -1, 0, 1, 2......$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{7}{2\sqrt{35}} g_n$$

$$y_{n+1} = \frac{7}{2} f_n + \frac{\sqrt{35}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 12x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 12y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue I, Jan 2019- Available at www.ijraset.com

Table: 1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	1	7
0	13	77
1	155	917
2	1847	10927
3	22009	130207

From the above table, we observe some interesting relations among the solutions which are presented below:

1) x_{n+1} and y_{n+1} are always odd.
2) Relations among the solutions

a) $x_{n+2} - 6x_{n+1} - y_{n+1} = 0$

b) $y_{n+2} - 6x_{n+2} + x_{n+1} = 0$

c) $y_{n+3} - 71x_{n+2} - 6x_{n+1} = 0$

d) $x_{n+3} - 12x_{n+2} + x_{n+1} = 0$

e) $x_{n+3} - 71x_{n+2} - y_{n+1} = 0$

f) $x_{n+3} - 6x_{n+2} - y_{n+2} = 0$

g) $x_{n+3} - 71x_{n+1} - 12y_{n+1} = 0$

h) $35x_{n+3} - 6y_{n+3} + y_{n+2} = 0$

i) $y_{n+1} - x_{n+2} + 6x_{n+1} = 0$

j) $12y_{n+1} - x_{n+3} - x_{n+1} = 0$

k) $12y_{n+1} - x_{n+3} + 71x_{n+1} = 0$

l) $2y_{n+2} - x_{n+3} + x_{n+1} = 0$

m) $12y_{n+3} - 71x_{n+3} + x_{n+1} = 0$

n) $349x_{n+1} - 72y_{n+1} + x_{n+3} = 0$

o) $349x_{n+2} - 781y_{n+1} + 6x_{n+3} = 0$

p) $349y_{n+2} - 4614y_{n+1} + 35x_{n+3} = 0$

q) $349y_{n+3} - 55019y_{n+1} + 420x_{n+3} = 0$

r) $6y_{n+1} - 71y_{n+2} + 35x_{n+3} = 0$

s) $6y_{n+3} - 35x_{n+3} - y_{n+2} = 0$

t) $y_{n+1} - 71y_{n+3} + 420x_{n+3} = 0$

u) $y_{n+3} - 12y_{n+2} + y_{n+1} = 0$

v) $280x_{n+1} - 3y_{n+3} + 353y_{n+1} = 0$

w) $y_{n+2} - 35x_{n+1} - 6y_{n+1} = 0$

x) $y_{n+3} - 420x_{n+1} - 71y_{n+1} = 0$

y) $6y_{n+3} - 35x_{n+1} - 71y_{n+2} = 0$

z) $y_{n+1} - 6x_{n+3} + 71x_{n+2} = 0$

aa) $y_{n+2} - x_{n+3} + 6x_{n+2} = 0$

bb) $y_{n+3} - 6x_{n+3} + x_{n+2} = 0$

cc) $y_{n+3} - 70x_{n+2} - y_{n+1} = 0$

dd) $6y_{n+2} - 35x_{n+1} - y_{n+1} = 0$

ee) $y_{n+3} - 35x_{n+2} + 6y_{n+2} = 0$

3) Each of the following expressions represents a nasty number

a) $6x_{2n+3} - 66x_{2n+2} + 12$

b) $\frac{1}{2}\left(x_{2n+4} - 131x_{2n+2} + 24\right)$

c) $6y_{2n+2} - 30x_{2n+2} + 12$

d) $y_{2n+3} - 65x_{2n+2} + 12$

e) $\frac{1}{71}\left(6y_{2n+4} - 4650x_{2n+2} + 852\right)$

f) $66x_{2n+4} - 786x_{2n+3} + 12$

g) $11y_{2n+2} - 5x_{2n+3} + 12$

h) $66y_{2n+3} - 390x_{2n+3} + 12$

i) $11y_{2n+4} - 775x_{2n+3} + 12$

j) $\frac{6}{71}\left(131y_{2n+2} - 5x_{2n+4} + 142\right)$

k) $131y_{2n+3} - 65x_{2n+4} + 12$

l) $786y_{2n+4} - 4650x_{2n+4} + 12$

m) $\frac{1}{7}\left(78y_{2n+2} - 6y_{2n+3} + 84\right)$

n) $\frac{1}{84}\left(930y_{2n+2} - 6y_{2n+4} + 1008\right)$

o) $\frac{1}{7}\left(930y_{2n+3} - 78y_{2n+4} + 84\right)$

4) Each of the following expressions represents a cubical integer

a) $x_{3n+4} - 11x_{3n+3} - 33x_{n+1} + 3x_{n+2}$

b) $\frac{1}{12}\left[x_{3n+5} - 131x_{3n+3} - 393x_{n+1} + 3x_{n+3}\right]$

c) $y_{3n+3} - 5x_{3n+3} - 15x_{n+1} + 3y_{n+1}$

d) $\frac{1}{6}\left[y_{3n+4} - 65x_{3n+3} - 195x_{n+1} + 3y_{n+2}\right]$

e) $\frac{1}{71}\left[y_{3n+5} - 775x_{3n+3} - 2325x_{n+1} + 3y_{n+3}\right]$

f) $11x_{3n+5} - 131x_{3n+4} - 393x_{n+2} + 33x_{n+3}$

g) $\frac{1}{6}\left[11y_{3n+3} - 5x_{3n+4} - 15x_{n+2} + 33y_{n+1}\right]$

h) $11y_{3n+4} - 65x_{3n+4} - 195x_{n+2} + 33y_{n+2}$

i) $\frac{1}{6}\left[11y_{3n+5} - 775x_{3n+4} - 2325x_{n+2} + 33y_{n+3}\right]$

j) $\frac{1}{497}\left[917y_{3n+3} - 35x_{3n+5} + 2751y_{n+1} - 105x_{n+3}\right]$

k) $\frac{1}{6}\left[131y_{3n+4} - 65x_{3n+5} - 195x_{n+3} + 393y_{n+2}\right]$

l) $131y_{3n+5} - 775x_{3n+5} - 2325x_{n+3} + 393y_{n+3}$

m) $\frac{1}{7}\left[13y_{3n+3} - y_{3n+4} + 39y_{n+1} - 3y_{n+2}\right]$

n) $\frac{1}{84}\left[155y_{3n+3} - y_{3n+5} + 465y_{n+1} - 3y_{n+3}\right]$

o) $\frac{1}{7}\left[155y_{3n+4} - 13y_{3n+5} + 465y_{n+2} - 39y_{n+3}\right]$

5) Each of the following expressions represents a bi-quadratic integer

a) $x_{4n+5} - 11x_{4n+4} - 44x_{2n+2} + 4x_{2n+3} + 6$

b) $\frac{1}{12}\left[x_{4n+6} - 131x_{4n+4} - 524x_{2n+2} + 4x_{2n+4} + 72\right]$

c) $y_{4n+4} - 5x_{4n+4} - 20x_{2n+2} + 4y_{2n+2} + 6$

d) $\frac{1}{6}\left[y_{4n+5} - 65x_{4n+4} - 260x_{2n+2} + 4y_{2n+3} + 36\right]$

e) $\frac{1}{71}\left[y_{4n+6} - 775x_{4n+4} - 3100x_{2n+2} + 4y_{2n+4} + 426\right]$

f) $11x_{4n+6} - 131x_{4n+5} - 524x_{2n+3} + 44x_{2n+4} + 6$

g) $\frac{1}{6}\left[11y_{4n+4} - 5x_{4n+5} - 20x_{2n+3} + 44y_{2n+2} + 36\right]$

h) $11y_{4n+5} - 65x_{4n+5} - 260x_{2n+3} + 44y_{2n+3} + 6$

i) $\frac{1}{6}\left[11y_{4n+6} - 775x_{4n+5} - 3100x_{2n+3} + 44y_{2n+4} + 36\right]$

j) $\frac{1}{71}\left[131y_{4n+4} - 5x_{4n+6} + 524y_{2n+2} - 20x_{2n+4} + 426\right]$

k) $\frac{1}{6}\left[131y_{4n+5} - 65x_{4n+6} - 260x_{2n+4} + 524y_{2n+3} + 36\right]$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue I, Jan 2019- Available at www.ijraset.com

l) $131y_{4n+6} - 775x_{4n+6} - 3100x_{2n+4} + 524y_{2n+4} + 6$

m) $\frac{1}{7}\left[13y_{4n+4} - y_{4n+5} + 52y_{2n+2} - 4y_{2n+3} + 42\right]$

n) $\frac{1}{84}\left[155y_{4n+4} - y_{4n+6} + 620y_{2n+2} - 4y_{2n+4} + 504\right]$

o) $\frac{1}{7}\left[155y_{4n+5} - 13y_{4n+6} + 620y_{2n+3} - 52y_{2n+4} + 42\right]$

6) Each of the following expressions represents a quintic integer

a) $x_{5n+6} - 11x_{5n+5} - 55x_{3n+3} + 5x_{3n+4} - 110x_{n+1} + 10x_{n+2}$

b) $\frac{1}{12}\left[x_{5n+7} - 131x_{5n+5} - 655x_{3n+3} + 5x_{3n+5} - 1310x_{n+1} + 10x_{n+3}\right]$

c) $y_{5n+5} - 5x_{5n+5} - 25x_{3n+3} + 5y_{3n+3} - 50x_{n+1} + 10y_{n+1}$

d) $\frac{1}{6}\left[y_{5n+6} - 65x_{5n+5} - 325x_{3n+3} + 5y_{3n+4} - 650x_{n+1} + 10y_{n+2}\right]$

e) $\frac{1}{71}\left[y_{5n+7} - 775x_{5n+5} - 3875x_{3n+3} + 5y_{3n+5} - 7750x_{n+1} + 10y_{n+3}\right]$

f) $11x_{5n+7} - 131x_{5n+6} - 655x_{3n+4} + 55x_{3n+5} - 1310x_{n+2} + 110x_{n+3}$

g) $\frac{1}{6}\left[11y_{5n+5} - 5x_{5n+6} - 25x_{3n+4} + 55y_{3n+3} - 50x_{n+2} + 110y_{n+1}\right]$

h) $11y_{5n+6} - 65x_{5n+6} - 325x_{3n+4} + 55y_{3n+4} - 650x_{n+2} + 110y_{n+2}$

i) $\frac{1}{6}\left[11y_{5n+7} - 775x_{5n+6} - 3875x_{3n+4} + 55y_{3n+5} - 7750x_{n+2} + 110y_{n+3}\right]$

j) $\frac{1}{71}\left[655y_{3n+3} - 524y_{5n+5} - 20x_{5n+7} - 25x_{3n+5} + 1965y_{n+1} - 75x_{n+3}\right]$

k) $\frac{1}{6}\left[131y_{5n+6} - 65x_{5n+7} - 325x_{3n+5} + 655y_{3n+4} - 650x_{n+3} + 1310y_{n+2}\right]$

l) $131y_{5n+7} - 775x_{5n+7} - 3875x_{3n+5} + 655y_{3n+5} - 7750x_{n+3} + 1310y_{n+3}$

m) $\frac{1}{7}\left[13y_{5n+5} - y_{5n+6} + 65y_{3n+3} - 5y_{3n+4} + 130y_{n+1} - 10y_{n+2}\right]$

n) $\frac{1}{84}\left[155y_{5n+5} - y_{5n+7} + 775y_{3n+3} - 5y_{3n+5} + 1550y_{n+1} - 10y_{n+3}\right]$

o) $\frac{1}{7}\left[155y_{5n+6} - 13y_{5n+7} + 775y_{3n+4} - 65y_{3n+5} + 1550y_{n+2} - 130y_{n+3}\right]$

B. Remarkable Observations

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue I, Jan 2019- Available at www.ijraset.com

Table: 2 Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$7X^2 - 5Y^2 = 28$	$(x_{n+2} - 11x_{n+1}, 13x_{n+1} - x_{n+2})$
2	$7X^2 - 5Y^2 = 4032$	$(x_{n+3} - 131x_{n+1}, 155x_{n+1} - x_{n+3})$
3	$7X^2 - 5Y^2 = 28$	$(y_{n+1} - 5x_{n+1}, 7x_{n+1} - y_{n+1})$
4	$7X^2 - 5Y^2 = 1008$	$(y_{n+2} - 65x_{n+1}, 77x_{n+1} - y_{n+2})$
5	$7X^2 - 5Y^2 = 141148$	$(y_{n+3} - 775x_{n+1}, 917x_{n+1} - y_{n+3})$
6	$7X^2 - 5Y^2 = 28$	$(11x_{n+3} - 131x_{n+2}, 155x_{n+2} - 13x_{n+3})$
7	$7X^2 - 5Y^2 = 1008$	$(11y_{n+1} - 5x_{n+2}, 7x_{n+2} - 13y_{n+1})$
8	$7X^2 - 5Y^2 = 28$	$(11y_{n+2} - 65x_{n+2}, 77x_{n+2} - 13y_{n+2})$
9	$7X^2 - 5Y^2 = 1008$	$(11y_{n+3} - 775x_{n+2}, 917x_{n+2} - 13y_{n+3})$
10	$7X^2 - 5Y^2 = 141148$	$(131y_{n+1} - 5x_{n+3}, 7x_{n+3} - 155y_{n+1})$
11	$7X^2 - 5Y^2 = 1008$	$(131y_{n+2} - 65x_{n+3}, 77x_{n+3} - 155y_{n+2})$
12	$7X^2 - 5Y^2 = 28$	$(131y_{n+3} - 775x_{n+3}, 917x_{n+3} - 155y_{n+3})$
13	$5X^2 - 7Y^2 = 980$	$(13y_{n+1} - y_{n+2}, y_{n+2} - 11y_{n+1})$
14	$5X^2 - 7Y^2 = 141120$	$(155y_{n+1} - y_{n+3}, y_{n+3} - 131y_{n+1})$
15	$5X^2 - 7Y^2 = 980$	$(155y_{n+2} - 13y_{n+3}, 11y_{n+3} - 131y_{n+2})$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue I, Jan 2019- Available at www.ijraset.com

2) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(X, Y)
1	$7X - 5Y^2 = 14$	$(x_{2n+3} - 11x_{2n+2}, 13x_{n+1} - x_{n+2})$
2	$84X - 5Y^2 = 2016$	$(x_{2n+4} - 131x_{2n+2}, 155x_{n+1} - x_{n+3})$
3	$7X - 5Y^2 = 14$	$(y_{2n+2} - 5x_{2n+2}, 7x_{n+1} - y_{n+1})$
4	$42X - 5Y^2 = 504$	$(y_{2n+3} - 65x_{2n+2}, 77x_{n+1} - y_{n+2})$
5	$497X - 5Y^2 = 70574$	$(y_{2n+4} - 775x_{2n+2}, 917x_{n+1} - y_{n+3})$
6	$7X - 5Y^2 = 14$	$(11x_{2n+4} - 131x_{2n+3}, 155x_{n+2} - 13x_{n+3})$
7	$42X - 5Y^2 = 504$	$(11y_{2n+2} - 5x_{2n+3}, 7x_{n+2} - 13y_{n+1})$
8	$7X - 5Y^2 = 14$	$(11y_{2n+3} - 65x_{2n+3}, 77x_{n+2} - 13y_{n+2})$
9	$42X - 5Y^2 = 504$	$(11y_{2n+4} - 775x_{2n+3}, 917x_{n+2} - 13y_{n+3})$
10	$497X - 5Y^2 = 70574$	$(131y_{2n+2} - 5x_{2n+4}, 7x_{n+3} - 155y_{n+1})$
11	$42X - 5Y^2 = 504$	$(131y_{2n+3} - 65x_{2n+4}, 77x_{n+3} - 155y_{n+2})$
12	$7X - 5Y^2 = 14$	$(131y_{2n+4} - 775x_{2n+4}, 917x_{n+3} - 155y_{n+3})$
13	$5X - Y^2 = 70$	$(13y_{2n+2} - y_{2n+3}, y_{n+2} - 11y_{n+1})$
14	$60X - Y^2 = 10080$	$(155y_{2n+2} - y_{2n+4}, y_{n+3} - 131y_{n+1})$
15	$5X - Y^2 = 70$	$(155y_{2n+3} - 13y_{2n+4}, 11y_{n+3} - 131y_{n+2})$

REFERENCES

[1] Dickson L.E., History of Theory of Numbers, Chelsea Publishing Company,Newyork,Vol.II,1952

[2] Mordell L.J., Diophantine Equations, Academic Press , Newyork,1969.

[3] Gopalan M.A, Vidhyalakshmi S, Sumathi G, Integral points on the Hyperbola, $x^2+6xy+y^2+40x+80y+40=0$ Bessel J Math., 2(3), 2012, 159-164.

[4] Gopalan M.A., Sumathi G., Vidhyalakshmi S., Observation on the Hyperbola, $y^2=24x^2+1$ Bessel J Math, 4, 2013, 21-25.

[5] Geetha T., Gopalan M.A., Vidhyalakshmi S., Observation on the hyperbola, $y^2=72x^2+1$, Scholars Journal of Physics, Mathematics and Statistics, 1(1), 2014, 1-13.

[6] Gopalan M.A, Vidhyalakshmi S., Umarani J., Remarkable Observations on the hyperbola $y^2=24x^2+1$, Bulletin of Mathematics and Statistics Research, 1, 2014, 9-12.

[7] Gopalan M.A , Vidhyalakshmi S, Premalatha E, Vanitha M, Observation on the hyperbola, $y^2=220x^2+9$, Star Research Journal, Vol.4, Issue3(2), 2016, 12-16.

[8] Ramya S, Kavitha A. On the positive pell equation, $y^2=90x^2+31$, Journal of Mathematics and informatics, 11(1), 2017, 11-17.

[9] Sumathi G., Observations on the hyperbola, $y^2=150x^2+16$, International Journal of Recent Trends in Engineering and Research, 3(9), 2017, 198-206.

[10] Meena K, Gopalan M.A., Sivaranjani V, On the positive pell equation $y^2=102x^2+33$. International Journal of Advanced Education and Research 2(1), 2017, 91-96

[11] Gopalan M.A, Devibala S., Swetha T., On the positive pell equation $y^2=5x^2+4$, IOSR Journal of Mathematics, Vol 13, Issue 5, Ver.1, 2017, 65-69.

[12] Usha Rani T.R, Dhivya K, On the positive pell equation $y^2=14x^2+18$, International Journal of Academic Research and Development, Volume 3, Issue 3, 2018, 43-50.