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Observation on the Negative Pell Equation $y^2 = 12x^2 - 23$

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***Abstract:** The binary quadratic equation represented by the negative pellian $y^2 = 12x^2 - 23$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas.*
***Keywords:** Binary quadratic, Hyperbola, Parabola, Pell equation, Integer solutions*

I. INTRODUCTION

The subject of Diophantine equation is one of the areas in Number Theory that has attracted many Mathematicians since antiquity and it has a long history. Obviously, the Diophantine equation are rich in variety [1-3]. In particular, the binary quadratic diophantine equation of the form $y^2 = Dx^2 - N\ (N > 0,\ D > 0\ \textit{and square free})$ is referred as the negative form of the pell equation (or) related pell equation. It is worth to observe that the negative pell equation is not always solvable. For example, the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solutions whereas $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [4-10] for a few negative pell equations with integer solutions. In this communication, the negative pell equation given by $y^2 = 12x^2 - 23$ is considered and analysed for its integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas.

II. METHOD OF ANALYSIS

The Negative Pell equation representing hyperbola under consideration is

$$y^2 = 12x^2 - 23 \tag{1}$$

The smallest positive integer solutions of (1) are

$$x_0 = 2,\ y_0 = 5$$

To obtain the other solutions of (1), consider the pellian equation

$$y^2 = 12x^2 + 1 \tag{2}$$

whose initial solution is given by

$$\tilde{x}_0 = 2,\ \tilde{y}_0 = 7$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by

$$\tilde{x}_n = \frac{1}{4\sqrt{3}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (7 + 4\sqrt{3})^{n+1} + (7 - 4\sqrt{3})^{n+1}$$

$$g_n = (7 + 4\sqrt{3})^{n+1} - (7 - 4\sqrt{3})^{n+1},\quad n = -1, 0, 1, \ldots$$

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions to (1) are given by

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$$x_{n+1} = f_n + \frac{5}{4\sqrt{3}} g_n$$

$$y_{n+1} = \frac{5}{2} f_n + 2\sqrt{3} g_n$$

The recurrence relation satisfied by the solution x and y are given by

$$x_{n+3} - 14x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 14y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x_n and y_n satisfying (1) are given in the Table: 1 below:

Table: 1 Numerical Examples

n	x_n	y_n
0	2	5
1	24	83
2	334	1157
3	4652	16115
4	64794	224453

From the above table, we observe some interesting relations among the solutions which are presented below:

1) x_n values are even.
2) y_n values are odd .

A. Relation Among The Solutions Are Given Below

1) $x_{n+1} = 97x_{n+3} - 28y_{n+3}$
2) $x_{n+2} = 7x_{n+1} - 2y_{n+1}$
3) $7x_{n+3} = 97x_{n+2} + 2y_{n+1}$
4) $7y_{n+2} = 24x_{n+2} + y_{n+1}$
5) $x_{n+3} = 97x_{n+1} + 28y_{n+1}$
6) $y_{n+1} = 97y_{n+3} - 336x_{n+3}$
7) $y_{n+2} = 24x_{n+1} + 7y_{n+1}$
8) $y_{n+3} = 336x_{n+1} + 97y_{n+1}$
9) $48x_{n+2} = y_{n+3} - y_{n+1}$
10) $14y_{n+2} = y_{n+3} + y_{n+1}$
11) $x_{n+1} = 14x_{n+2} - x_{n+3}$
12) $x_{n+1} = 7x_{n+2} - 2y_{n+2}$
13) $7x_{n+1} = 97x_{n+2} - 2y_{n+3}$
14) $7x_{n+3} = x_{n+2} + 2y_{n+3}$
15) $7y_{n+2} = y_{n+3} - 24x_{n+2}$
16) $x_{n+1} = x_{n+3} - 4y_{n+2}$
17) $7y_{n+1} = 97y_{n+2} - 24x_{n+3}$
18) $7x_{n+2} = x_{n+3} - 2y_{n+2}$
19) $24x_{n+1} = 7y_{n+3} - 97y_{n+2}$
20) $24x_{n+3} = 7y_{n+3} - y_{n+2}$

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B. Each Of The Following Expressions Represents A Nasty Number

1) $\frac{6}{23}[48x_{2n+2} - 10y_{2n+2} + 46]$

2) $\frac{6}{23}[83x_{2n+2} - 5x_{2n+3} + 46]$

3) $\frac{3}{161}[1157x_{2n+2} - 5x_{2n+4} + 644]$

4) $\frac{12}{161}[288x_{2n+2} - 5y_{2n+3} + 161]$

5) $\frac{6}{2231}[8016x_{2n+2} - 10y_{2n+4} + 4462]$

6) $\frac{6}{161}[48x_{2n+3} - 166y_{2n+2} + 322]$

7) $\frac{12}{2231}[24x_{2n+4} - 1157y_{2n+2} + 2231]$

8) $\frac{1}{23}[12y_{2n+3} - 144y_{2n+2} + 276]$

9) $\frac{6}{161}[y_{2n+4} - 167y_{2n+2} + 322]$

10) $\frac{6}{23}[1157x_{2n+3} - 83x_{2n+4} + 46]$

11) $\frac{12}{23}[288x_{2n+3} - 83y_{2n+3} + 23]$

12) $\frac{12}{161}[4008x_{2n+3} - 83y_{2n+4} + 161]$

13) $\frac{12}{161}[288x_{2n+4} - 1157y_{2n+3} + 161]$

14) $\frac{12}{23}[4008x_{2n+4} - 1157y_{2n+4} + 23]$

15) $\frac{1}{23}[144y_{2n+4} - 2004y_{2n+3} + 276]$

C. Each Of The Following Expressions Represents A Cubical Integer

1) $\frac{1}{23}[48x_{3n+3} - 10y_{3n+3} + 144x_{n+1} - 30y_{n+1}]$

2) $\frac{1}{23}[83x_{3n+3} - 5x_{3n+4} + 249x_{n+1} - 15x_{n+2}]$

3) $\frac{1}{322}[1157x_{3n+3} - 5x_{3n+5} + 3471x_{n+1} - 15x_{n+3}]$

4) $\frac{2}{161}[288x_{3n+3} - 5y_{3n+4} + 864x_{n+1} - 15y_{n+2}]$

5) $\frac{1}{2231}[8016x_{3n+3} - 10y_{3n+5} + 24048x_{n+1} - 30y_{n+3}]$

6) $\frac{1}{161}[48x_{3n+4} - 166y_{3n+3} + 144x_{n+2} - 498y_{n+1}]$

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7) $\frac{2}{2231}\left[24x_{3n+5} - 1157y_{3n+3} + 72x_{n+3} - 3471y_{n+1}\right]$

8) $\frac{1}{23}\left[2y_{3n+4} - 24y_{3n+3} + 6y_{n+2} - 72y_{n+1}\right]$

9) $\frac{1}{161}\left[y_{3n+4} - 167y_{3n+3} + 3y_{n+3} - 501y_{n+1}\right]$

10) $\frac{1}{23}\left[1157x_{3n+4} - 83x_{3n+5} + 3471x_{n+2} - 249x_{n+3}\right]$

11) $\frac{2}{23}\left[288x_{3n+4} - 83y_{3n+4} + 864x_{n+2} - 249y_{n+2}\right]$

12) $\frac{2}{161}\left[4008x_{3n+4} - 83y_{3n+5} + 12024x_{n+2} - 249y_{n+3}\right]$

13) $\frac{2}{161}\left[288x_{3n+5\backslash} - 1157y_{3n+4} + 864x_{n+3} - 3471y_{n+2}\right]$

14) $\frac{2}{23}\left[4008x_{3n+5\backslash} - 1157y_{3n+5} + 12024x_{n+3} - 3471y_{n+3}\right]$

15) $\frac{2}{23}\left[12y_{3n+5\backslash} - 167y_{3n+4} + 36y_{n+3} - 501y_{n+2}\right]$

D. Each Of The Following Expressions Represents A Biquadratic Integer

1) $\frac{1}{23}\left[48x_{4n+4} - 10y_{4n+4} + 192x_{2n+2} - 40y_{2n+2} + 138\right]$

2) $\frac{1}{23}\left[83x_{4n+4} - 5x_{4n+5} - 20x_{2n+3} + 332x_{2n+2} + 138\right]$

3) $\frac{1}{322}\left[1157_{4n+4} - 5x_{4n+6} - 20x_{2n+4} + 4628x_{2n+2} + 1932\right]$

4) $\frac{2}{161}\left[288x_{4n+4} - 5y_{4n+5} - 20y_{2n+3} + 1152x_{2n+2} + 966\right]$

5) $\frac{1}{2231}\left[8016x_{4n+4} - 10y_{4n+6} - 40y_{2n+4} + 32064x_{2n+2} + 13386\right]$

6) $\frac{1}{161}\left[48x_{4n+5} - 166y_{4n+4} - 664y_{2n+2} + 192x_{2n+3} + 966\right]$

7) $\frac{2}{2231}\left[24x_{4n+6} - 1157y_{4n+4} - 4628y_{2n+2} + 96x_{2n+4} + 6693\right]$

8) $\frac{2}{23}\left[y_{4n+5} - 12y_{4n+4} - 48y_{2n+2} + 4y_{2n+3} + 69\right]$

9) $\frac{1}{161}\left[y_{4n+6} - 167y_{4n+4} - 668y_{2n+2} + 4y_{2n+4} + 966\right]$

10) $\frac{1}{23}\left[1157x_{4n+5} - 83x_{4n+6} - 332x_{2n+4} + 4628x_{2n+3} + 138\right]$

11) $\frac{2}{23}\left[288x_{4n+5} - 83y_{4n+5} - 332y_{2n+3} + 1152x_{2n+3} + 69\right]$

12) $\frac{2}{161}\left[4008x_{4n+5} - 83y_{4n+6} - 332y_{2n+4} + 16032x_{2n+3} + 483\right]$

13) $\frac{2}{161}\left[288x_{4n+6} - 1157y_{4n+5} - 4628y_{2n+3} + 1152x_{2n+4} + 483\right]$

14) $\frac{2}{23}\left[4008x_{4n+6} - 1157y_{4n+6} - 4628y_{2n+4} + 16032x_{2n+4} + 69\right]$

15) $\frac{2}{23}\left[12y_{4n+6} - 167y_{4n+5} - 668y_{2n+3} + 48y_{2n+4} + 69\right]$

E. Each Of The Following Expressions Represents A Quintic Integer

1) $\frac{1}{23}\left[48x_{5n+5} - 10y_{5n+5} + 240x_{3n+3} - 50y_{3n+3} + 480x_{n+1} - 100y_{n+1}\right]$

2) $\frac{1}{23}\left[83x_{5n+5} - 5x_{5n+6} + 415x_{3n+3} - 25x_{3n+4} + 830x_{n+1} - 50x_{n+2}\right]$

3) $\frac{1}{322}\left[1157x_{5n+5} - 5x_{5n+7} + 5785x_{3n+3} - 25x_{3n+5} + 11570x_{n+1} - 50x_{n+3}\right]$

4) $\frac{2}{161}\left[288x_{5n+5} - 5y_{5n+6} + 1440x_{3n+3} - 25y_{3n+4} + 2880x_{n+1} - 50y_{n+2}\right]$

5) $\frac{1}{2231}\left[8016x_{5n+5} - 10y_{5n+7} + 40080x_{3n+3} - 50y_{3n+5} + 80160x_{n+1} - 100y_{n+3}\right]$

6) $\frac{1}{161}\left[48x_{5n+6} - 166y_{5n+5} + 240x_{3n+4} - 830y_{3n+3} + 480x_{n+2} - 1660y_{n+1}\right]$

7) $\frac{2}{2231}\left[24x_{5n+7} - 1157y_{5n+5} + 120x_{3n+5} - 5785y_{3n+3} + 240x_{n+3} - 11570y_{n+1}\right]$

8) $\frac{2}{23}\left[y_{5n+6} - 12y_{5n+5} + 5y_{3n+4} - 60y_{3n+3} + 10y_{n+2} - 120y_{n+1}\right]$

9) $\frac{1}{161}\left[y_{5n+7} - 167y_{5n+5} + 5y_{3n+5} - 835y_{3n+3} + 10y_{n+3} - 1670y_{n+1}\right]$

10) $\frac{1}{23}\left[1157x_{5n+6} - 83x_{5n+7} + 5785x_{3n+4} - 415x_{3n+5} + 11570x_{n+2} - 830x_{n+3}\right]$

11) $\frac{2}{23}\left[288x_{5n+6} - 83y_{5n+6} + 1440x_{3n+4} - 415y_{3n+4} + 2880x_{n+2} - 830y_{n+2}\right]$

12) $\frac{2}{161}\left[4008x_{5n+6} - 83y_{5n+7} + 20040x_{3n+4} - 415y_{3n+5} + 40080x_{n+2} - 830y_{n+3}\right]$

13) $\frac{2}{161}\left[288x_{5n+7} - 1157y_{5n+6} + 1440x_{3n+5} - 5785y_{3n+4} + 2880x_{n+3} - 11570y_{n+2}\right]$

14) $\frac{2}{23}\left[4008x_{5n+7} - 1157y_{5n+7} + 20040x_{3n+5} - 5785y_{3n+5} + 40080x_{n+3} - 11570y_{n+3}\right]$

15) $\frac{2}{23}\left[12y_{5n+7} - 167y_{5n+6} + 60y_{4n+6} - 835y_{4n+5} + 120y_{n+3} - 1670y_{n+2}\right]$

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III.REMARKABLE OBSERVATIONS

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table: 2 below:

Table: 2 Hyperbolas

S.NO	Hyperbola	(X,Y)
1	$Y^2-12X^2=2116$	$(4y_{n+1}-10x_{n+1}\ ,\ 48x_{n+1}-10y_{n+1})$
2	$Y^2-12X^2=2116$	$(2x_{n+2}-24x_{n+1}\ ,\ 83x_{n+1}-5x_{n+2})$
3	$Y^2-12X^2=414736$	$(2x_{n+3}-334x_{n+1}\ ,1157x_{n+1}-5x_{n+3})$
4	$Y^2-3X^2=25921$	$(4y_{n+2}-166x_{n+1}\ ,\ 288x_{n+1}-5y_{n+2})$
5	$Y^2-12X^2=19909444$	$(2314x_{n+1}-4y_{n+3}\ ,8016x_{n+1}-10y_{n+3})$
6	$Y^2-12X^2=103684$	$(48y_{n+1}-10x_{n+2}\ ,\ 48x_{n+2}-166y_{n+1})$
7	$Y^2-12X^2=4977361$	$(334y_{n+1}-5x_{n+3}\ ,\ 1157y_{n+1}-24x_{n+3})$
8	$Y^2-3X^2=76176$	$(83y_{n+1}-5y_{n+2}\ ,12y_{n+2}-144y_{n+1})$
9	$Y^2-3X^2=14930496$	$(1157y_{n+1}-5y_{n+3}\ ,12y_{n+3}-2004y_{n+1})$
10	$Y^2-24X^2=2116$	$(12x_{n+3}-167x_{n+2}\ ,\ 1157x_{n+2}-83x_{n+3})$
11	$Y^2-12X^2=529$	$(24y_{n+2}-83x_{n+2}\ ,\ 288x_{n+2}-83y_{n+2})$
12	$Y^2-12X^2=25921$	$(24y_{n+3}-1157x_{n+2}\ ,\ 4008x_{n+2}-83y_{n+3})$
13	$Y^2-12X^2=25921$	$(334y_{n+2}-83x_{n+3}\ ,\ 288x_{n+3}-1157y_{n+2})$
14	$Y^2-12X^2=529$	$(334y_{n+3}-1157x_{n+3}\ ,\ 4008x_{n+3}-1157y_{n+3})$
15	$Y^2-3X^2=76176$	$(1157y_{n+2}-83y_{n+3}\ ,\ 144y_{n+3}-2004y_{n+2})$

2) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table: 3 below:

Table: 3 Parabolas

S.NO	Parabola	(X,Y)
1	$23Y-12X^2=2116$	$(4y_{n+1}-10x_{n+1}\ ,\ 48x_{2n+2}-10y_{2n+2}+46)$
2	$23Y-12X^2=2116$	$(2x_{n+1}-24x_{n+1}\ ,83x_{2n+2}-5x_{2n+3}+46)$
3	$322Y-12X^2=414736$	$(2x_{n+3}-334x_{n+1}\ ,1157x_{2n+2}-5x_{2n+4}+644)$
4	$322Y-12X^2=103684$	$(4y_{n+2}-166x_{n+1}\ ,288x_{2n+2}-5y_{2n+3}+161)$
5	$2231Y-12X^2=19909444$	$(2314x_{n+1}-4y_{n+3}\ ,8016y_{2n+2}-10x_{2n+4}+4462)$
6	$161Y-12X^2=103684$	$(48y_{n+1}-10x_{n+2}\ ,\ 48x_{2n+3}-166y_{2n+2}+322)$
7	$4462Y-48X^2=19909444$	$(334y_{n+1}-5x_{n+3}\ ,24x_{2n+4}-1157y_{2n+2}+2231)$
8	$46Y-X^2=25392$	$(83y_{n+1}-5y_{n+2}\ ,12y_{2n+3}-144y_{2n+2}+276)$
9	$644Y-X^2=4976832$	$(1157y_{n+1}-5y_{n+3}\ ,12y_{2n+4}-2004y_{2n+2}+3864)$
10	$23Y-24X^2=2116$	$(12x_{n+3}-167x_{n+2}\ ,1157x_{2n+3}-83x_{2n+4}+46)$
11	$23Y-24X^2=1058$	$(24y_{n+2}-83x_{n+2}\ ,\ 288x_{2n+3}-83y_{2n+3}+23)$
12	$161Y-24X^2=51842$	$(24y_{n+3}-1157x_{n+2}\ ,\ 4008x_{2n+3}-83y_{2n+4}+161)$
13	$161Y-24X^2=51842$	$(334y_{n+2}-83x_{n+3}\ ,\ 288x_{2n+4}-1157y_{2n+3}+161)$
14	$23Y-24X^2=1058$	$(334y_{n+3}-1157x_{n+3}\ ,\ 4008x_{2n+4}-1157y_{2n+4}+23)$
15	$46Y-X^2=25392$	$(1157y_{n+2}-83y_{n+3}\ ,\ 144y_{2n+4}-2004y_{2n+3}+276)$

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