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On The Negative Pell Equation $y^2 = 15x^2 - 14$

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***Abstract:** The binary quadratic equation represented by the negative pellian $y^2 = 15x^2 - 14$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.*
***Keywords:** Binary quadratic, hyperbola, parabola, pell equation, integral solutions.*

I. INTRODUCTION

Diophantine equation of the form $y^2 = Dx^2 - 1$, where $D > 0$ and square free, is known as negative pell equation. In general, the general form of negative pell equation is represented by $y^2 = Dx^2 - N$, $N > 0$, $D > 0$ and square free. It is known that negative pell equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solutions whereas $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. It is observed that the negative pell equation do not always have integer solutions. For negative pell equations with integer solutions, one may refer [1-12].

In this communication, yet another negative pell equation given by $y^2 = 7x^2 - 14$ is considered for its non-zero distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

II. METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 15x^2 - 14 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 3,\ y_0 = 11$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 15x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{15}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(4+\sqrt{15}\right)^{n+1} + \left(4-\sqrt{15}\right)^{n+1}$$

$$g_n = \left(4+\sqrt{15}\right)^{n+1} - \left(4-\sqrt{15}\right)^{n+1},\quad n = -1, 0, 1, 2......$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{3}{2} f_n + \frac{11}{2\sqrt{15}} g_n$$

$$y_{n+1} = \frac{11}{2} f_n + \frac{3}{2}\sqrt{15} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 8x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 8y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

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Table: 1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	3	11
0	23	89
1	181	701
2	1425	5519
3	11219	43451
4	88327	342089

From the above table, we observe some interesting relations among the solutions which are presented below:

1) x_{n+1} and y_{n+1} are always odd
2) Relations among the solutions

a) $4x_{n+1} - x_{n+2} + y_{n+1} = 0$
b) $x_{n+1} - 4x_{n+2} + y_{n+2} = 0$
c) $4x_{n+1} - 31x_{n+2} + y_{n+3} = 0$
d) $31x_{n+1} - x_{n+3} + 8y_{n+1} = 0$
e) $x_{n+1} - 31x_{n+3} + 8y_{n+3} = 0$
f) $15x_{n+1} + 4y_{n+1} - y_{n+2} = 0$
g) $120x_{n+1} + 31y_{n+1} - y_{n+3} = 0$
h) $15x_{n+1} + 31y_{n+2} - 4y_{n+3} = 0$
i) $31x_{n+2} - 4x_{n+3} + y_{n+1} = 0$
j) $4x_{n+2} - x_{n+3} + y_{n+2} = 0$
k) $x_{n+2} - 4x_{n+3} + y_{n+3} = 0$
l) $15x_{n+2} + y_{n+1} - 4y_{n+2} = 0$
m) $30x_{n+2} - y_{n+3} + y_{n+1} = 0$
n) $15x_{n+2} + 4y_{n+2} - y_{n+3} = 0$
o) $15x_{n+1} + 4y_{n+2} - y_{n+3} = 0$
p) $15\,x_{n+3} + 4y_{n+1} - 31y_{n+2} = 0$
q) $120x_{n+3} + y_{n+1} - 31y_{n+3} = 0$
r) $15x_{n+3} + y_{n+2} - 4y_{n+3} = 0$
s) $x_{n+1} - x_{n+3} + 2y_{n+2} = 0$

3) Each of the following expressions represents a nasty number

a) $\frac{6}{7}\left(89x_{2n+2} - 11x_{2n+3} + 14\right)$
b) $\frac{3}{28}\left(701x_{2n+2} - 11x_{2n+4} + 112\right)$
c) $\frac{6}{7}\left(45x_{2n+2} - 11y_{2n+2} + 14\right)$
d) $\frac{6}{28}\left(345x_{2n+2} - 11y_{2n+3} + 56\right)$
e) $\frac{6}{217}\left(2715x_{2n+2} - 11y_{2n+4} + 434\right)$

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f) $\frac{6}{7}\left(701x_{2n+3}-89x_{2n+4}+14\right)$

g) $\frac{3}{14}\left(45x_{2n+3}-89y_{2n+2}+56\right)$

h) $\frac{6}{7}\left(345x_{2n+3}-89y_{2n+3}+14\right)$

i) $\frac{3}{14}\left(2715x_{2n+3}-89y_{2n+4}+56\right)$

j) $\frac{6}{217}\left(45x_{2n+4}-701y_{2n+2}+434\right)$

k) $\frac{3}{14}\left(345x_{2n+4}-701y_{2n+3}+56\right)$

l) $\frac{6}{7}\left(2715x_{2n+4}-701y_{2n+4}+14\right)$

m) $\frac{6}{7}\left(3y_{2n+3}-23y_{2n+2}+14\right)$

n) $\frac{3}{28}\left(3y_{2n+4}-181y_{2n+2}+112\right)$

o) $\frac{6}{7}\left(23y_{2n+4}-181y_{2n+3}+14\right)$

4) Each of the following expressions represents a cubical integer

a) $\frac{1}{7}\left[267x_{n+1}-33x_{n+2}+89x_{3n+3}-11x_{3n+4}\right]$

b) $\frac{1}{56}\left[2103x_{n+1}-33x_{n+3}+701x_{3n+3}-11x_{3n+5}\right]$

c) $\frac{1}{7}\left[135x_{n+1}-33y_{n+1}+45x_{3n+3}-11y_{3n+3}\right]$

d) $\frac{1}{28}\left[1035x_{n+1}-33y_{n+2}+345x_{3n+3}-11y_{3n+4}\right]$

e) $\frac{1}{217}\left[8145x_{n+1}-33y_{n+3}+2715x_{3n+3}-11y_{3n+5}\right]$

f) $\frac{1}{7}\left[2103x_{n+2}-267x_{n+3}+701x_{3n+4}-89x_{3n+5}\right]$

g) $\frac{1}{28}\left[135x_{n+2}-267y_{n+1}+45x_{3n+4}-89y_{3n+3}\right]$

h) $\frac{1}{7}\left[1035x_{n+2}-267y_{n+2}+345x_{3n+4}-89y_{3n+4}\right]$

i) $\frac{1}{28}\left[8145x_{n+2}-267y_{n+3}+2715x_{3n+4}-89y_{3n+5}\right]$

j) $\frac{1}{217}\left[135x_{n+3}-2103y_{n+1}+45x_{3n+5}-701y_{3n+3}\right]$

k) $\frac{1}{28}\left[1035x_{n+3}-2103y_{n+2}+345x_{3n+5}-701y_{3n+4}\right]$

l) $\frac{1}{7}\left[8145x_{n+3}-2103y_{n+3}+2715x_{3n+5}-701y_{3n+5}\right]$

m) $\frac{1}{7}\left[9y_{n+2}-69y_{n+1}+3y_{3n+4}-23y_{3n+3}\right]$

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n) $\frac{1}{56}\left[9y_{n+3}-543y_{n+1}+3y_{3n+5}-181y_{3n+3}\right]$

o) $\frac{1}{7}\left[69y_{n+3}-543y_{n+2}+23y_{3n+5}-181y_{3n+4}\right]$

5) Each of the following expressions represents a bi-quadratic integer

a) $\frac{1}{7}\left[89x_{4n+4}-11x_{4n+5}+356x_{2n+2}-44x_{2n+3}+42\right]$

b) $\frac{1}{56}\left[701x_{4n+4}-11x_{4n+6}+2804x_{2n+2}-44x_{2n+4}+336\right]$

c) $\frac{1}{7}\left[45x_{4n+4}-11y_{4n+4}+180x_{2n+2}-44y_{2n+2}+42\right]$

d) $\frac{1}{28}\left[345x_{4n+4}-11y_{4n+5}+1380x_{2n+2}-44y_{2n+3}+168\right]$

e) $\frac{1}{217}\left[2715x_{4n+4}-11y_{4n+6}+10860x_{2n+2}-44y_{2n+4}+1302\right]$

f) $\frac{1}{7}\left[701x_{4n+5}-89x_{4n+6}+2804x_{2n+3}-356x_{2n+4}+42\right]$

g) $\frac{1}{28}\left[45x_{4n+5}-89y_{4n+4}+180x_{2n+3}-356y_{2n+2}+168\right]$

h) $\frac{1}{7}\left[345x_{4n+5}-89y_{4n+5}+1380x_{2n+3}-356y_{2n+3}+42\right]$

i) $\frac{1}{28}\left[2715x_{4n+5}-89y_{4n+6}+10860x_{2n+3}-356y_{2n+4}+168\right]$

j) $\frac{1}{217}\left[45x_{4n+6}-701y_{4n+4}+180x_{2n+4}-2804y_{2n+2}+1302\right]$

k) $\frac{1}{28}\left[345x_{4n+6}-701y_{4n+5}+1380x_{2n+4}-2804y_{2n+3}+168\right]$

l) $\frac{1}{7}\left[2715x_{4n+6}-701y_{4n+6}+10860x_{n+3}-2804y_{n+3}+42\right]$

m) $\frac{1}{7}\left[3y_{4n+5}-23y_{4n+4}+12y_{2n+3}-92y_{2n+2}+42\right]$

n) $\frac{1}{56}\left[3y_{4n+6}-181y_{4n+4}+12y_{2n+4}-724y_{2n+2}+336\right]$

o) $\frac{1}{7}\left[23y_{4n+6}-181y_{4n+5}+92y_{2n+4}-724y_{2n+3}+42\right]$

6) Each of the following expressions represents a quintic integer

a) $\frac{1}{7}\left[890x_{n+1}-110x_{n+2}+445x_{3n+3}-55x_{3n+4}+89x_{5n+5}-11x_{5n+6}\right]$

b) $\frac{1}{56}\left[7010x_{n+1}-110x_{n+3}+3505x_{3n+3}-55x_{3n+5}+701x_{5n+5}-11x_{5n+7}\right]$

c) $\frac{1}{7}\left[450x_{n+1}-110y_{n+1}+225x_{3n+3}-55y_{3n+3}+45x_{5n+5}-11y_{5n+5}\right]$

d) $\frac{1}{28}\left[3450x_{n+1}-110y_{n+2}+1725x_{3n+3}-55y_{3n+4}+345x_{5n+5}-11y_{5n+6}\right]$

e) $\frac{1}{217}\left[27150x_{n+1}-110y_{n+3}+13575x_{3n+3}-55y_{3n+5}+2715x_{5n+5}-11y_{5n+7}\right]$

f) $\frac{1}{7}\left[7010x_{n+2}-890x_{n+3}+3505x_{3n+4}-445x_{3n+5}+701x_{5n+6}-89x_{5n+7}\right]$

g) $\frac{1}{28}\left[450x_{n+2}-890y_{n+1}+225x_{3n+4}-445y_{3n+3}+45x_{5n+6}-89y_{5n+5}\right]$

h) $\frac{1}{7}\left[3450x_{n+2}-890y_{n+2}+1725x_{3n+4}-445y_{3n+4}+345x_{5n+6}-89y_{5n+6}\right]$

i) $\frac{1}{28}\left[27150x_{n+2}-890y_{n+3}+13575x_{3n+4}-445y_{3n+5}+2715x_{5n+6}-89y_{5n+7}\right]$

j) $\frac{1}{217}\left[450x_{n+3}-7010y_{n+1}+225x_{3n+5}-3505y_{3n+3}+45x_{5n+7}-701y_{5n+5}\right]$

k) $\frac{1}{28}\left[3450x_{n+3}-7010y_{n+2}+1725x_{3n+5}-3505y_{3n+4}+345x_{5n+7}-701y_{5n+6}\right]$

l) $\frac{1}{7}\left[27150x_{n+3}-7010y_{n+3}+13575x_{3n+5}-3505y_{3n+5}+2715x_{5n+7}-701y_{5n+7}\right]$

m) $\frac{1}{7}\left[30y_{n+2}-230y_{n+1}+15y_{3n+4}-115y_{3n+3}+3y_{5n+6}-23y_{5n+5}\right]$

n) $\frac{1}{56}\left[30y_{n+3}-1810y_{n+1}+15y_{3n+5}-905y_{3n+3}+3y_{5n+7}-181y_{5n+5}\right]$

o) $\frac{1}{7}\left[230y_{n+3}-1810y_{n+2}+115y_{3n+5}-905y_{3n+4}+23y_{5n+7}-181y_{5n+6}\right]$

III.REMARKABLE OBSERVATIONS

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbola	(X, Y)
1	$X^2-15Y^2=196$	$(89x_{n+1}-11x_{n+2}, 3x_{n+2}-23x_{n+1})$
2	$X^2-15Y^2=12544$	$(701x_{n+1}-11x_{n+3}, 3x_{n+3}-181x_{n+1})$
3	$X^2-15Y^2=196$	$(45x_{n+1}-11y_{n+1}, 3y_{n+1}-11x_{n+1})$
4	$X^2-15Y^2=3136$	$(345x_{n+1}-11y_{n+2}, 3y_{n+2}-89x_{n+1})$
5	$X^2-15Y^2=188356$	$(2715x_{n+1}-11y_{n+3}, 3y_{n+3}-701x_{n+1})$
6	$X^2-15Y^2=196$	$(701x_{n+2}-89x_{n+3}, 23x_{n+3}-181x_{n+2})$
7	$X^2-15Y^2=3136$	$(45x_{n+2}-89y_{n+1}, 23y_{n+1}-11x_{n+2})$
8	$X^2-15Y^2=196$	$(345x_{n+2}-89y_{n+2}, 23y_{n+2}-89x_{n+2})$
9	$X^2-15Y^2=3136$	$(2715x_{n+2}-89y_{n+3}, 23y_{n+3}-701x_{n+2})$
10	$X^2-15Y^2=188356$	$(45x_{n+3}-701y_{n+1}, 181y_{n+1}-11x_{n+3})$

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11	$X^2 - 15Y^2 = 3136$	$(345x_{n+3} - 701y_{n+2}, 181y_{n+2} - 89x_{n+3})$
12	$X^2 - 15Y^2 = 196$	$(2715x_{n+3} - 701y_{n+3}, 181y_{n+3} - 701x_{n+3})$
13	$735X^2 - 49Y^2 = 144060$	$(3y_{n+2} - 23y_{n+1}, 89y_{n+1} - 11y_{n+2})$
14	$47040X^2 - 3136Y^2 = 5990069760$	$(3y_{n+3} - 181y_{n+1}, 701y_{n+1} - 11y_{n+3})$
15	$735X^2 - 49Y^2 = 144060$	$(23y_{n+3} - 181y_{n+2}, 701y_{n+2} - 89y_{n+3})$

2) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabola	(X, Y)
1	$7X - 15Y^2 = 98$	$(89x_{2n+2} - 11x_{2n+3}, 3x_{n+2} - 23x_{n+1})$
2	$56X - 15Y^2 = 6272$	$(701x_{2n+2} - 11x_{2n+4}, 3x_{n+3} - 181x_{n+1})$
3	$7X - 15Y^2 = 98$	$(45x_{2n+2} - 11y_{2n+2}, 3y_{n+1} - 11x_{n+1})$
4	$28X - 15Y^2 = 1568$	$(345x_{2n+2} - 11y_{2n+3}, 3y_{n+2} - 89x_{n+1})$
5	$217X - 15Y^2 = 94178$	$(2715x_{2n+2} - 11y_{2n+4}, 3y_{n+3} - 701x_{n+1})$
6	$7X - 15Y^2 = 98$	$(701x_{2n+3} - 89x_{2n+4}, 23x_{n+3} - 181x_{n+2})$
7	$28X - 15Y^2 = 1568$	$(45x_{2n+3} - 89y_{2n+2}, 23y_{n+1} - 11x_{n+2})$
8	$7X - 15Y^2 = 98$	$(345x_{2n+3} - 89y_{2n+3}, 23y_{n+2} - 89x_{n+2})$
9	$28X - 15Y^2 = 1568$	$(2715x_{2n+3} - 89y_{2n+4}, 23y_{n+3} - 701x_{n+2})$
10	$217X - 15Y^2 = 94178$	$(45x_{2n+4} - 701y_{2n+2}, 181y_{n+1} - 11x_{n+3})$
11	$28X - 15Y^2 = 1568$	$(345x_{2n+4} - 701y_{2n+3}, 181y_{n+2} - 89x_{n+3})$
12	$7X - 15Y^2 = 98$	$(2715x_{2n+4} - 701y_{2n+4}, 181y_{n+3} - 701x_{n+3})$
13	$735X - 7Y^2 = 10290$	$(3y_{2n+3} - 23y_{2n+2}, 89y_{n+1} - 11y_{n+2})$
14	$47040X - 56Y^2 = 5268480$	$(3y_{2n+4} - 181y_{2n+2}, 701y_{n+1} - 11y_{n+3})$
15	$735X - 7Y^2 = 10290$	$(23y_{2n+4} - 181y_{2n+3}, 701y_{n+2} - 89y_{n+3})$

IV. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $y^2 = 15x^2 - 14$. As the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

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