International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue III, Mar 2019- Available at www.ijraset.com

# Observation on the Binary Quadratic Equation $y^2 = 105x^2 + 4^t, t \geq 0$

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***Abstract: The binary quadratic equation is considered and a few interesting properties among the solutions are presented.***
***Keywords: Binary quadratic, integral solutions, Generalized Fibonacci sequences, Generalized Lucas Sequences.***

## I. INTRODUCTION

The Binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is a non-square positive integer has been studied by various Mathematicians for its non-trivial integral solutions when D takes different integral values [1-5]. In this context one may also refer [4,5]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation representing a hyperbola. A few interesting properties among the solutions are presented.

## II. NOTATIONS

1) $t_{m,n}$ : Polygonal number of rank $n$ with size $m$
2) $P_n^m$ : Pyramidal number of rank $n$ with size $m$
3) $\Pr_n$ : Pronic number of rank $n$
4) $S_n$ : Star number of rank $n$
5) $Ct_{m,n}$ : Centered Pyramidal number of rank $n$ with size $m$
6) $GF_n(k,s)$ : Generalized Fibonacci sequence number of rank $n$
7) $GL_n(k,s)$ : Generalized Lucas sequence number of rank $n$

## III.METHOD OF ANALYSIS

The binary non-homogeneous quadratic Diophantine equation represents a hyperbola to be solved for its non-zero integral solutions is

$$y^2 = 105x^2 + 4^t, t \geq 0 \tag{1}$$

The smallest positive integer solution $(x_0, y_0)$ of (1) is

$$x_0 = 4(2^t), y_0 = 41(2^t) \tag{2}$$

To obtain the other solutions of (1), consider the pellian equation

$$y^2 = 105x^2 + 1 \tag{3}$$

Applying the Brahmagupta lemma between the solutions $(x_0, y_0)$ and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{4(2^t)}{2} f_n + \frac{41(2^t)}{2\sqrt{105}} g_n$$

$$y_{n+1} = \frac{41(2^t)}{2} f_n + \frac{420(2^t)}{2\sqrt{105}} g_n$$

*A. A Few Numerical Examples Are Given In The Following Table 1*

Tabe1: Examples

| $n$ | $x_{n+1}$ | $y_{n+1}$ |
|---|---|---|
| -1 | $4(2^t)$ | $41(2^t)$ |
| 0 | $328(2^t)$ | $3361(2^t)$ |
| 1 | $26892(2^t)$ | $275561(2^t)$ |
| 2 | $2204816(2^t)$ | $22592641(2^t)$ |

*B. A Few Interesting Properties Are Given Below*

*1)* The recurrence relations satisfied by the values of $x_{n+1}$ and $y_{n+1}$ are respectively.

$2x_{n+3} - 164x_{n+2} + 2x_{n+1} = 0$, $n = -1,0,1.......$

$2y_{n+3} - 164y_{n+2} + 2y_{n+1} = 0$, $n = -1,0,1.......$

A few interesting relations among the solutions are given below:

*a)* $x_{n+3} - 82x_{n+2} + x_{n+1} = 0$

*b)* $4y_{n+1} - x_{n+2} + 41x_{n+1} = 0$

*c)* $4y_{n+2} - 41x_{n+2} + x_{n+1} = 0$

*d)* $4y_{n+3} - 3361x_{n+2} + 41x_{n+1} = 0$

*e)* $328y_{n+1} - x_{n+3} + 3361x_{n+1} = 0$

*f)* $8y_{n+2} - x_{n+3} + x_{n+1} = 0$

*g)* $328y_{n+3} - 3361x_{n+3} + x_{n+1} = 0$

*h)* $y_{n+2} - 41y_{n+1} - 420x_{n+1} = 0$

*i)* $y_{n+3} - 3361y_{n+1} - 34440x_{n+1} = 0$

*j)* $41y_{n+3} - 3361y_{n+2} - 420x_{n+1} = 0$

*k)* $4y_{n+1} - 41x_{n+3} + 3361x_{n+2} = 0$

*l)* $4y_{n+2} - x_{n+3} + 41x_{n+2} = 0$

*m)* $4y_{n+3} - 41x_{n+3} + x_{n+2} = 0$

*n)* $41y_{n+2} - y_{n+1} - 420x_{n+2} = 0$

*o)* $y_{n+3} - 41y_{n+2} - 420x_{n+2} = 0$

*p)* $3361y_{n+2} - 41y_{n+1} - 420x_{n+3} = 0$

*q)* $3361y_{n+3} - y_{n+1} - 34440x_{n+3} = 0$

*r)* $y_{n+3} - y_{n+1} - 840x_{n+2} = 0$

*s)* $41y_{n+3} - y_{n+2} - 420x_{n+3} = 0$

*t)* $y_{n+1} - y_{n+3} - 82y_{n+2} = 0$

2) *Each Of The Following Is A Nasty Number*

a) $\frac{1}{(2^t)}\left[123x_{2n+3}-10083x_{2n+2}+12(2^t)\right]$

b) $\frac{1}{2(2^t)}\left[3x_{2n+4}-20163x_{2n+2}+8(2^t)\right]$

c) $\frac{1}{(2^t)}\left[492y_{2n+2}-5040x_{2n+2}+12(2^t)\right]$

d) $\frac{1}{(2^t)}\left[12y_{2n+3}-10080x_{2n+2}+12(2^t)\right]$

e) $\frac{1}{3361(2^t)}\left[492y_{2n+4}-33883920x_{2n+2}+40332(2^t)\right]$

f) $\frac{1}{(2^t)}\left[10083x_{2n+4}-826683x_{2n+3}+12(2^t)\right]$

g) $\frac{1}{41(2^t)}\left[40332y_{2n+2}-5040x_{2n+3}+492(2^t)\right]$

h) $\frac{1}{(2^t)}\left[40332y_{2n+3}-68880x_{2n+3}+12(2^t)\right]$

i) $\frac{1}{41(2^t)}\left[40332y_{2n+4}-33883920x_{2n+3}+492(2^t)\right]$

j) $\frac{1}{3361(2^t)}\left[3306732y_{2n+2}-5040x_{2n+4}+40332(2^t)\right]$

k) $\frac{1}{41(2^t)}\left[3306732y_{2n+3}-413280x_{2n+4}+492(2^t)\right]$

l) $\frac{1}{(2^t)}\left[3306732y_{2n+4}-33883920x_{2n+4}+12(2^t)\right]$

m) $\frac{1}{35(2^t)}\left[34440y_{2n+2}-420y_{2n+3}+420(2^t)\right]$

n) $\frac{1}{41(2^t)}\left[40338y_{2n+2}-6y_{2n+4}+492(2^t)\right]$

o) $\frac{1}{(2^t)}\left[80676y_{2n+3}-164y_{2n+4}+12(2^t)\right]$

3) *Each Of The Following Is A Cubical Integer*

a) $\frac{1}{2(2^t)}\left[41x_{3n+4}-3361x_{3n+3}+123x_{n+2}-10083x_{n+1}\right]$

b) $\frac{1}{4(2^t)}\left[x_{3n+5}-6721x_{3n+3}+3x_{n+3}-20163x_{n+1}\right]$

c) $\frac{1}{(2^t)}\left[82y_{3n+3}-840x_{3n+3}+246y_{n+1}-2520x_{n+1}\right]$

d) $\frac{1}{(2^t)}\left[2y_{3n+4}-1680x_{2n+2}+6y_{n+2}-5040x_{n+1}\right]$

e) $\frac{1}{3361(2^t)}\left[82y_{3n+5}-5647320x_{3n+3}+246y_{n+3}-16941960x_{n+1}\right]$

f) $\frac{1}{2(2^t)}[3361x_{3n+5} - 275561x_{3n+4} + 10083x_{n+3} - 826683x_{n+2}]$

g) $\frac{1}{41(2^t)}[6722y_{3n+3} - 840x_{3n+4} + 20166y_{n+1} - 2520x_{n+2}]$

h) $\frac{1}{2(2^t)}[6722y_{3n+4} - 68880x_{3n+4} + 20166y_{n+2} - 206640x_{n+2}]$

i) $\frac{1}{41(2^t)}[6722y_{3n+5} - 647320x_{3n+4} + 20166y_{n+3} - 16941960x_{n+2}]$

j) $\frac{1}{3361(2^t)}[551122y_{3n+3} - 840x_{3n+5} + 1653366y_{n+1} - 2520x_{n+3}]$

k) $\frac{1}{41(2^t)}[551122y_{3n+4} - 68880x_{3n+5} + 1653366y_{n+2} - 206640x_{n+3}]$

l) $\frac{1}{(2^t)}[551122y_{3n+5} - 5647320x_{3n+5} + 1653366y_{n+3} - 16941960x_{n+3}]$

m) $\frac{1}{210(2^t)}[34440y_{3n+3} - 420y_{3n+4} + 103320y_{n+1} - 1260y_{n+2}]$

n) $\frac{1}{41(2^t)}[6723y_{3n+3} - y_{3n+5} + 20169y_{n+1} - 3y_{n+3}]$

o) $\frac{1}{(2^t)}[13446y_{3n+4} - 164x_{3n+5} + 40338y_{n+2} - 492y_{n+3}]$

4) Each of the following expression is a biquadratic integer

a) $\frac{1}{2(2^t)}[41x_{4n+5} - 3361x_{4n+4} + 164x_{2n+3} - 13444x_{2n+4} + 12(2^t)]$

b) $\frac{1}{4(2^t)}[x_{4n+6} - 6721x_{4n+4} + 4x_{2n+4} - 26884x_{2n+2} + 24(2^t)]$

c) $\frac{1}{(2^t)}[82y_{4n+4} - 840x_{4n+4} + 328y_{2n+2} - 3360x_{2n+2} + 6(2^t)]$

d) $\frac{1}{(2^t)}[2y_{4n+5} - 1680x_{4n+4} + 8y_{2n+3} - 6720x_{2n+2} + 6(2^t)]$

e) $\frac{1}{3361(2^t)}[82y_{4n+6} - 5647320x_{4n+4} + 328y_{2n+4} - 22589280x_{2n+2} + 20166(2^t)]$

f) $\frac{1}{2(2^t)}[3361x_{4n+6} - 275561x_{4n+5} + 13444x_{2n+4} - 1102244x_{2n+3} + 12(2^t)]$

g) $\frac{1}{41(2^t)}[6722y_{4n+4} - 840x_{4n+5} + 26888y_{2n+2} - 3360x_{2n+3} + 246(2^t)]$

h) $\frac{1}{(2^t)}[6722y_{4n+5} - 68880x_{4n+5} + 26888y_{2n+3} - 275520x_{2n+3} + 6(2^t)]$

i) $\frac{1}{41(2^t)}[6722y_{4n+6} - 56473270x_{4n+5} + 26888y_{2n+4} - 22589280x_{2n+3} + 246(2^t)]$

j) $\frac{1}{3361(2^t)}[551122y_{4n+4} - 840x_{4n+6} + 2204488y_{2n+2} - 33604x_{2n+4} + 20166(2^t)]$

k) $\frac{1}{41(2^t)}[551122y_{4n+5} - 68880x_{4n+6} + 2204488y_{2n+3} - 275520x_{2n+4} + 246(2^t)]$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue III, Mar 2019- Available at www.ijraset.com

l) $\frac{1}{(2^t)}\left[551122y_{4n+6} - 5647320x_{4n+6} + 2204488y_{2n+4} - 22589280x_{2n+4} + 6(2^t)\right]$

m) $\frac{1}{210(2^t)}\left[34440y_{4n+4} - 420y_{4n+5} + 137760y_{2n+2} - 1680y_{2n+3} + 1260(2^t)\right]$

n) $\frac{1}{41(2^t)}\left[6723y_{4n+4} - y_{4n+6} + 26892y_{2n+2} - 4y_{2n+4} + 246(2^t)\right]$

o) $\frac{1}{(2^t)}\left[13446y_{4n+5} - 164y_{4n+6} + 53784y_{2n+3} - 656y_{2n+4} + 6(2^t)\right]$

5) Each of the following expression is a quintic integer

a) $\frac{1}{2(2^t)}\begin{bmatrix} 41x_{5n+6} - 3361x_{5n+5} + 205x_{3n+4} - 16805x_{3n+3} + 615x_{n+2} \\ -50415x_{n+1} - 205x_{n+2} + 16805x_{n+1} \end{bmatrix}$

b) $\frac{1}{4(2^t)}\begin{bmatrix} x_{5n+7} - 6721x_{5n+5} + 5x_{3n+5} - 33605x_{3n+3} + 15x_{n+3} \\ -100815x_{n+1} - 5x_{n+3} + 33605x_{n+1} \end{bmatrix}$

c) $\frac{1}{(2^t)}\begin{bmatrix} 82y_{5n+5} - 840x_{5n+5} + 410y_{3n+3} - 4200x_{3n+3} + 1230y_{n+1} \\ -12600x_{n+1} - 410y_{n+1} + 4200x_{n+1} \end{bmatrix}$

d) $\frac{1}{(2^t)}\begin{bmatrix} 2y_{5n+6} - 1680x_{5n+5} + 10y_{3n+4} - 8400x_{2n+2} + 30y_{n+2} \\ -25200x_{n+1} - 10y_{n+2} + 8400x_{n+1} \end{bmatrix}$

e) $\frac{1}{3361(2^t)}\begin{bmatrix} 82y_{5n+7} - 5647320x_{5n+5} + 410y_{3n+5} - 28236600x_{3n+3} + 1230y_{n+3} \\ -84709800x_{n+1} - 410y_{n+3} + 28236600x_{n+1} \end{bmatrix}$

f) $\frac{1}{2(2^t)}\begin{bmatrix} 3361x_{5n+7} - 275561x_{5n+6} + 16805x_{3n+5} - 1377805x_{3n+4} + 50415x_{n+3} \\ -4133415x_{n+2} - 16805x_{n+3} + 1377805x_{n+2} \end{bmatrix}$

g) $\frac{1}{41(2^t)}\begin{bmatrix} 6722y_{5n+5} - x_{5n+6} + 19355y_{3n+3} - 3225x_{3n+4} + 58065y_{n+1} \\ -9675x_{n+2} - 193555y_{n+1} + 3225x_{n+2} \end{bmatrix}$

h) $\frac{1}{(2^t)}\begin{bmatrix} 6722y_{5n+6} - 68880x_{5n+6} + 33610y_{3n+4} - 344400x_{3n+4} + 100830y_{n+2} \\ -100830y_{n+2} - 33610y_{n+2} + 344400x_{n+2} \end{bmatrix}$

i) $\frac{1}{41(2^t)}\begin{bmatrix} 6722y_{5n+7} - 5647320x_{5n+6} + 33610y_{3n+5} - 28236600x_{3n+4} + 100830y_{n+3} \\ -84709800x_{n+2} - 33610y_{n+3} + 28236600x_{n+2} \end{bmatrix}$

j) $\frac{1}{3361(2^t)}\begin{bmatrix} 551122y_{5n+5} - 840x_{5n+7} + 2755610y_{3n+3} - 4200x_{3n+5} + 8266830y_{n+1} \\ -12600x_{n+3} - 2755610y_{n+1} + 4200x_{n+3} \end{bmatrix}$

k) $\frac{1}{41(2^t)}\begin{bmatrix} 551122y_{5n+6} - 68880x_{5n+7} + 2755610y_{3n+4} - 344400x_{3n+5} + 8266830y_{n+2} \\ -1033200x_{n+3} - 2755610y_{n+2} + 34440x_{n+3} \end{bmatrix}$

l) $\frac{1}{(2^t)}\begin{bmatrix} 551122y_{5n+7} - 5647320x_{5n+7} + 27556100y_{3n+5} - 28236600x_{3n+5} + 8266830y_{n+3} \\ -84709800x_{n+3} - 2755610y_{n+3} + 2823660x_{n+3} \end{bmatrix}$

m) $\frac{1}{210(2^t)}\begin{bmatrix} 34440y_{5n+5} - 420y_{5n+6} + 172200y_{3n+3} - 2100y_{3n+4} + 516600y_{n+1} \\ -6300y_{n+2} - 172200y_{n+1} + 2100y_{n+2} \end{bmatrix}$

n) $\frac{1}{41(2^t)}\begin{bmatrix} 6723y_{5n+5} - y_{5n+7} + 33615y_{3n+3} - 5y_{3n+5} + 100845y_{n+1} \\ -15y_{n+3} - 33615y_{n+1} + 5y_{n+3} \end{bmatrix}$

o) $\frac{1}{(2^t)}\begin{bmatrix} 13446y_{5n+6} - 164y_{5n+7} + 67230y_{3n+4} - 820y_{3n+5} + 201690y_{n+2} \\ -2460y_{n+3} - 67230y_{n+2} + 820y_{n+3} \end{bmatrix}$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue III, Mar 2019- Available at www.ijraset.com

*6)* Remarkable Observations

*a)* Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of hyperbola which are presented in table below.

| S.No | Hyperbola | $(X_n, Y_n)$ |
|---|---|---|
| 1 | $X_n^2 - 420Y_n^2 = 16(4^t)$ | $\begin{pmatrix} 41x_{n+2} - 3361x_{n+1} \\ 164x_{n+1} - 2x_{n+2} \end{pmatrix}$ |
| 2 | $1681X_n^2 - 1680Y_n^2 = 107584(4^t)$ | $\begin{pmatrix} x_{n+3} - 6721x_{n+1} \\ 6723x_{n+1} - x_{n+3} \end{pmatrix}$ |
| 3 | $X_n^2 - 105Y_n^2 = 4(4^t)$ | $\begin{pmatrix} 82y_{n+1} - 840x_{n+1} \\ 82x_{n+1} - 8y_{n+1} \end{pmatrix}$ |
| 4 | $1681X_n^2 - 105Y_n^2 = 6724(4^t)$ | $\begin{pmatrix} 2y_{n+2} - 1680x_{n+1} \\ 6722x_{n+1} - 8y_{n+2} \end{pmatrix}$ |
| 5 | $X_n^2 - 105Y_n^2 = 45185284(4^t)$ | $\begin{pmatrix} 82y_{n+3} - 5647320x_{n+1} \\ 551122x_{n+1} - 8y_{n+3} \end{pmatrix}$ |
| 6 | $X_n^2 - 420Y_n^2 = 16(4^t)$ | $\begin{pmatrix} 3361x_{n+3} - 275561x_{n+2} \\ 13446x_{n+2} - 164x_{n+3} \end{pmatrix}$ |
| 7 | $X_n^2 - 105Y_n^2 = 6724(4^t)$ | $\begin{pmatrix} 6722y_{n+1} - 840x_{n+2} \\ 82x_{n+2} - 656y_{n+1} \end{pmatrix}$ |
| 8 | $X_n^2 - 105Y_n^2 = 4(4^t)$ | $\begin{pmatrix} 6722y_{n+2} - 68880x_{n+2} \\ 6722x_{n+2} - 656y_{n+2} \end{pmatrix}$ |
| 9 | $X_n^2 - 105Y_n^2 = 6724(4^t)$ | $\begin{pmatrix} 6722y_{n+3} - 5647320x_{n+2} \\ 551122x_{n+2} - 656y_{n+3} \end{pmatrix}$ |
| 10 | $X_n^2 - 105Y_n^2 = 45185284(4^t)$ | $\begin{pmatrix} 551122y_{n+1} - 840x_{n+3} \\ 82x_{n+3} - 53784y_{n+1} \end{pmatrix}$ |
| 11 | $X_n^2 - 105Y_n^2 = 6724(4^t)$ | $\begin{pmatrix} 551122y_{n+2} - 68880x_{n+3} \\ 6722x_{n+3} - 53784y_{n+2} \end{pmatrix}$ |
| 12 | $X_n^2 - 105Y_n^2 = 4(4^t)$ | $\begin{pmatrix} 551122y_{n+3} - 5647320x_{n+3} \\ 551122x_{n+3} - 53784y_{n+3} \end{pmatrix}$ |
| 13 | $X_n^2 - 105Y_n^2 = 176400(4^t)$ | $\begin{pmatrix} 34440y_{n+1} - 420y_{n+2} \\ 41y_{n+2} - 3361y_{n+1} \end{pmatrix}$ |
| 14 | $35280X_n^2 - 35301Y_n^2 = 237222720(4^t)$ | $\begin{pmatrix} 6723y_{n+1} - y_{n+3} \\ y_{n+3} - 6721y_{n+1} \end{pmatrix}$ |
| 15 | $8820X_n^2 - 21Y_n^2 = 35280(4^t)$ | $\begin{pmatrix} 13446y_{n+2} - 164y_{n+3} \\ 3361y_{n+3} - 275561y_{n+2} \end{pmatrix}$ |

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ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue III, Mar 2019- Available at www.ijraset.com

*b)* Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of Parabola which are presented in table below.

| S.No | Parabola | $(X_n, Y_n)$ |
|---|---|---|
| 1 | $(2^t)X_n - 210Y_n^2 = 4(4^t)$ | $\begin{pmatrix} 41x_{2n+3} - 3361x_{2n+2} \\ 164x_{2n+2} - 2x_{2n+3} \end{pmatrix}$ |
| 2 | $1681(2^t)X_n - 420Y_n^2 = 13448(4^t)$ | $\begin{pmatrix} x_{2n+4} - 6721x_{2n+2} \\ 6723x_{2n+2} - x_{2n+4} \end{pmatrix}$ |
| 3 | $(2^t)X_n - 105Y_n^2 = 4(4^t)$ | $\begin{pmatrix} 82y_{2n+2} - 840x_{2n+2} \\ 82x_{2n+2} - 68y_{2n+2} \end{pmatrix}$ |
| 4 | $1681(2^t)X_n - 105Y_n^2 = 3362(4^t)$ | $\begin{pmatrix} 2y_{2n+3} - 1680x_{2n+2} \\ 6722x_{2n+2} - 8y_{2n+3} \end{pmatrix}$ |
| 5 | $3361(2^t)X_n - 105Y_n^2 = 22592642(4^t)$ | $\begin{pmatrix} 82y_{2n+4} - 5647320x_{2n+2} \\ 551122x_{2n+2} - 8y_{2n+4} \end{pmatrix}$ |
| 6 | $(2^t)X_n - 210Y_n^2 = 4(4^t)$ | $\begin{pmatrix} 6722y_{2n+2} - 840x_{2n+3} \\ 82x_{2n+3} - 656x_{2n+2} \end{pmatrix}$ |
| 7 | $41(2^t)X_n - 105Y_n^2 = 3362(4^t)$ | $\begin{pmatrix} 6722y_{2n+2} - 840x_{2n+3} \\ 82x_{2n+3} - 656y_{2n+2} \end{pmatrix}$ |
| 8 | $(2^t)X_n - 105Y_n^2 = 2(4^t)$ | $\begin{pmatrix} 6722y_{2n+3} - 68880x_{2n+3} \\ 6722x_{2n+3} - 656y_{2n+3} \end{pmatrix}$ |
| 9 | $41(2^t)X_n - 105Y_n^2 = 3362(4^t)$ | $\begin{pmatrix} 6722y_{2n+4} - 5647320x_{2n+3} \\ 551122x_{2n+3} - 656y_{2n+4} \end{pmatrix}$ |
| 10 | $3361(2^t)X_n - 105Y_n^2 = 22592642(4^t)$ | $\begin{pmatrix} 551122y_{2n+2} - 840x_{2n+4} \\ 82x_{2n+4} - 53784y_{2n+2} \end{pmatrix}$ |
| 11 | $41(2^t)X_n - 105Y_n^2 = 3362(4^t)$ | $\begin{pmatrix} 551122y_{2n+3} - 68880x_{2n+4} \\ 6722x_{2n+4} - 53784y_{2n+3} \end{pmatrix}$ |
| 12 | $(2^t)X_n - 105Y_n^2 = 2(4^t)$ | $\begin{pmatrix} 551122y_{2n+4} - 5647320x_{2n+4} \\ 551122x_{2n+4} - 53784y_{2n+4} \end{pmatrix}$ |
| 13 | $210(2^t)X_n - 105Y_n^2 = 176400(4^t)$ | $\begin{pmatrix} 34440y_{2n+2} - 420y_{2n+3} \\ 41y_{2n+3} - 3361y_{2n+2} \end{pmatrix}$ |
| 14 | $35280(2^t)X_n - 861Y_n^2 = 2892960(4^t)$ | $\begin{pmatrix} 6723y_{2n+2} - y_{2n+4} \\ y_{2n+4} - 6721y_{2n+2} \end{pmatrix}$ |
| 15 | $8820(2^t)X_n - 21Y_n^2 = 17640(4^t)$ | $\begin{pmatrix} 13446y_{2n+3} - 164y_{2n+4} \\ 3361y_{2n+4} - 275561y_{2n+3} \end{pmatrix}$ |

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 6.887
Volume 7 Issue III, Mar 2019- Available at www.ijraset.com

*c)* Employing the linear combination among the solutions of (1), one may generate integer solutions for other choices of straight line which are presented in the table below:

| S.No. | Straight line | $(Y, X)$ |
|---|---|---|
| 1 | $Y = 2X$ | $Y = x_{n+3} - 6721x_{n+1}$<br>$X = 41x_{n+2} - 3361x_{n+1}$ |
| 2 | $Y = X$ | $Y = 82y_{n+1} - 840x_{n+1}$<br>$X = 2y_{n+2} - 1680x_{n+1}$ |
| 3 | $Y = 41X$ | $Y = 6722y_{n+1} - 840x_{n+2}$<br>$X = 6722y_{n+2} - 68880x_{n+2}$ |
| 4 | $Y = 3361X$ | $Y = 82y_{n+3} - 1680x_{n+1}$<br>$X = 551122y_{n+3} - 5647320x_{n+3}$ |
| 5 | $Y = X$ | $Y = 41x_{n+2} - 3361x_{n+1}$<br>$X = 3361x_{n+3} - 275561x_{n+2}$ |
| 6 | $Y = 210X$ | $Y = 34440y_{n+1} - 420y_{n+2}$<br>$X = 82y_{n+1} - 840x_{n+1}$ |
| 7 | $Y = 210X$ | $Y = 34440y_{n+1} - 420y_{n+2}$<br>$X = 13446y_{n+2} - 164y_{n+3}$ |
| 8 | $Y = X$ | $Y = 6722y_{n+3} - 5647320x_{n+2}$<br>$X = 6723y_{n+1} - y_{n+3}$ |
| 9 | $Y = 2X$ | $Y = 3361x_{n+3} - 275561x_{n+2}$<br>$X = 2y_{n+2} - 1680x_{n+1}$ |
| 10 | $Y = 2X$ | $Y = 176y_{n+1} - 2y_{n+2}$<br>$X = 3361x_{n+3} - 275561x_{n+2}$ |
| 11 | $Y = 2X$ | $Y = x_{n+3} - 6721x_{n+1}$<br>$X = 551122y_{n+3} - 5647320x_{n+3}$ |
| 12 | $Y = X$ | $Y = 82y_{n+1} - 840x_{n+1}$<br>$X = 13446y_{n+2} - 164y_{n+3}$ |
| 13 | $Y = 41X$ | $Y = 6723y_{n+1} - y_{n+3}$<br>$X = 2y_{n+2} - 1680x_{n+1}$ |
| 14 | $Y = 3361X$ | $Y = 82y_{n+3} - 5647320x_{n+1}$<br>$X = 13446y_{n+2} - 164y_{n+3}$ |
| 15 | $Y = X$ | $Y = 551122y_{n+2} - 68880x_{n+3}$<br>$X = 6723y_{n+1} - y_{n+3}$ |

*d)* The solutions in terms of special integer sequence namely, generalized Fibonacci sequence $GF_n(k,s)$ and Lucas sequence $GL_n(k,s)$ are exhibited below.

$$x_{n+1} = 2(2^t)GL_{n+1}(82,-1) + 164(2^t)GF_{n+1}(82,-1)$$

$$y_{n+1} = \frac{41}{2}(2^t)GL_{n+1}(82,-1) + 1680(2^t)GF_{n+1}(82,-1)$$

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*e)* Employing the solutions (1) ,each of the following among the special polygonal, pyramidal, star numbers, pronic numbers is a congruent to under modulo 4

*i)* $\left(\frac{12p_y^5}{s_{y+1}-1}\right)^2 - 105\left(\frac{3p_x^3}{t_{3,x+1}}\right)^2$

*ii)* $\left(\frac{4p_y^5}{ct_{4,y}-1}\right)^2 - 105\left(\frac{12p_x^5}{s_{x+1}-1}\right)^2$

*iii)* $\left(\frac{3p_{y-2}^3}{t_{3,y-2}}\right)^2 - 105\left(\frac{4p_x^5}{ct_{4,x}-1}\right)^2$

*iv)* $\left(\frac{6p_y^3}{pr_{y+1}}\right)^2 - 105\left(\frac{p_x^5}{t_{3,x}}\right)^2$

*v)* $\left(\frac{4p_y^5}{ct_{4,x}-1}\right)^2 - 105\left(\frac{6p_{x-1}^4}{t_{3,2x-1}}\right)^2$

*vi)* $\left(\frac{36p_{y-2}^3}{s_{y+1}-1}\right)^2 - 105\left(\frac{3p_{x-2}^3}{t_{3,y-2}}\right)^2$

*vii)* $\left(\frac{12p_y^5}{s_{y+1}-1}\right)^2 - 105\left(\frac{3p_{x-2}^3}{t_{3,x-2}}\right)^2$

*viii)* $\left(\frac{4pt_{y-3}}{p_{y-3}^3}\right)^2 - 105\left(\frac{2p_{x-1}^5}{t_{4,x-1}}\right)^2$

*ix)* $\left(\frac{6p_y^5}{ct_{6,y}-1}\right)^2 - 105\left(\frac{3p_{x+1}^4 - p_{x+1}^3}{t_{4,x+1}}\right)^2$

*x)* $\left(\frac{6p_{y-1}^4}{t_{3,2y-1}}\right)^2 - 105\left(\frac{p_x^5}{t_{3,x}}\right)^2$

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