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On the Positive Pell Equation y2 = 21x2 + 4 $y^2 = 21x^2 + 4$

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***Abstract:** The binary quadratic Diophantine equation represented by the positive pellian* $y^2 = 21x^2 + 4$ *is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions we have obtained the solutions of other choices of hyperbolas and parabolas.*

***Keywords:** Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.*

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4].For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 21x^2 + 4$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. NOTATIONS

$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$ polygonal number of rank n with size m

$p^m = \frac{1}{6} n(n+1)((m-2)n + 5 - m)$ Pyramidal number of rank n with size m

III. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 21x^2 + 4 \qquad (1)$$

The smallest positive integer solutions of (1) are,

$$x_0 = 1, y_0 = 5 \quad D = 21$$

consider the pellian equation is

$$y^2 = 21x^2 + 1 \qquad (2)$$

The initial solution of pellian equation is

$$\tilde{x}_0 = 12, \tilde{y}_0 = 55,$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{21}} g_n, \; \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (55 + 12\sqrt{21})^{n+1} + (55 - 12\sqrt{21})^{n+1}$$

$$g_n = (55 + 12\sqrt{21})^{n+1} - (55 - 12\sqrt{21})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{21}} g_n$$

$$y_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{21}}{2} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 110x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 110y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x_n and y_n satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	1	5
1	115	527
2	12649	57965
3	1391275	6375623
4	153027601	701260565

From the above table, we observe some interesting relations among the solutions which are presented below,

Both x_n and y_n values are odd.

A. Each Of The Following Expression Is A Nasty Number

1) $\frac{1}{4}[5x_{2n+3} - 527x_{2n+2} + 48]$

2) $\frac{1}{176}[2x_{2n+4} - 23186x_{2n+2} + 2112]$

3) $\frac{1}{2}[30y_{2n+2} - 126x_{2n+2} + 24]$

4) $\frac{1}{11}[3y_{2n+3} - 2949x_{2n+2} + 132]$

5) $\frac{1}{6049}[15y_{2n+4} - 796887x_{2n+2} + 72588]$

6) $\frac{1}{4}[527x_{2n+4} - 57965x_{2n+3} + 48]$

7) $\frac{1}{55}[1581y_{2n+2} - 63x_{2n+3} + 660]$

8) $\frac{1}{2}[3162y_{2n+3} - 14490x_{2n+3} + 12]$

9) $\frac{1}{55}[1581y_{2n+4} - 796887x_{2n+3} + 660]$

10) $\frac{1}{6049}[173895 y_{2n+2} - 63 x_{2n+4} + 72588]$

11) $\frac{1}{55}[173895 y_{2n+3} - 7245 x_{2n+4} + 660]$

12) $\frac{1}{2}[347790 y_{2n+4} - 1593774 x_{2n+4} + 24]$

13) $\frac{1}{24}[690 y_{2n+2} - 6 y_{2n+3} + 288]$

14) $\frac{1}{1320}[37947 y_{2n+2} - 3 y_{2n+4} + 15840]$

15) $\frac{1}{12}[12649 y_{2n+3} - 115 y_{2n+4} + 48]$

B. Each Of The Following Expressions Is A Cubical Integer

1) $\frac{1}{48}[10 x_{3n+4} - 1054 x_{3n+3} + 30 x_{n+2} - 3162 x_{n+1}]$

2) $\frac{1}{528}[x_{3n+5} - 11593 x_{3n+3} + 3 x_{n+3} - 34779 x_{n+1}]$

3) $\frac{1}{2}[5 y_{3n+3} - 21 x_{3n+3} + 15 y_{n+1} - 63 x_{n+1}]$

4) $\frac{1}{22}[y_{3n+4} - 483 x_{3n+3} + 3 y_{n+2} - 1449 x_{n+1}]$

5) $\frac{1}{12098}[5 y_{3n+5} - 265629 x_{3n+3} + 15 y_{n+3} - 796887 x_{n+1}]$

6) $\frac{1}{24}[527 x_{3n+5} - 57965 x_{3n+4} + 1581 x_{n+3} - 173895 x_{n+2}]$

7) $\frac{1}{110}[527 y_{3n+3} - 21 x_{3n+4} + 1581 y_{n+1} - 63 x_{n+2}]$

8) $\frac{1}{2}[527 y_{3n+4} - 2415 x_{3n+4} + 1581 y_{n+2} - 7245 x_{n+2}]$

9) $\frac{1}{110}[527 y_{3n+5} - 265629 x_{3n+4} + 1581 y_{n+3} - 796887 x_{n+2}]$

10) $\frac{1}{12098}[57965 y_{3n+3} - 21 x_{3n+5} + 173895 y_{n+1} - 63 x_{n+3}]$

11) $\frac{1}{110}[57965 y_{3n+4} - 2415 x_{3n+5} + 173895 y_{n+2} - 7245 x_{n+3}]$

12) $\frac{1}{2}[57965 y_{3n+5} - 265629 x_{3n+5} + 173895 y_{n+3} - 796887 x_{n+3}]$

13) $\frac{1}{24}[115 y_{3n+3} - y_{3n+4} + 345 y_{n+1} - 3 y_{n+2}]$

14) $\frac{1}{2640}\left[12649y_{3n+3} - y_{3n+5} + 37947y_{n+1} - 3y_{n+3}\right]$

15) $\frac{1}{24}\left[12649y_{3n+4} - 115y_{3n+5} + 37947y_{n+2} - 345y_{n+3}\right]$

C. Each Of The Following Expressions Is A Bi-Quadratic Integer

1) $\frac{1}{48}\left[10x_{4n+5} - 1054x_{4n+4} + 40x_{2n+3} - 4216x_{2n+2} + 288\right]$

2) $\frac{1}{528}\left[x_{4n+6} - 11593x_{4n+4} + 4x_{2n+4} - 46372x_{2n+2} + 3168\right]$

3) $\frac{1}{2}\left[5y_{4n+4} - 21x_{4n+4} + 20y_{2n+2} - 84x_{2n+2} + 12\right]$

4) $\frac{1}{22}\left[y_{4n+5} - 483x_{4n+4} + 4y_{2n+3} - 1932x_{2n+2} + 132\right]$

5) $\frac{1}{12098}\left[5y_{4n+6} - 265629x_{4n+4} + 20y_{2n+4} - 1062516x_{2n+2} + 72588\right]$

6) $\frac{1}{24}\left[527x_{4n+6} - 57965x_{4n+5} + 2108x_{2n+4} - 231860x_{2n+3} + 144\right]$

7) $\frac{1}{110}\left[527y_{4n+4} - 21x_{4n+5} + 2108y_{2n+2} - 84x_{2n+3} + 660\right]$

8) $\frac{1}{2}\left[527y_{4n+5} - 2415x_{4n+5} + 2108y_{2n+3} - 9660x_{2n+3} + 12\right]$

9) $\frac{1}{110}\left[527y_{4n+6} - 265629x_{4n+5} + 2108y_{2n+4} - 1062516x_{2n+3} + 660\right]$

10) $\frac{1}{12098}\left[57965y_{4n+4} - 21x_{4n+6} + 231860y_{2n+2} - 84x_{2n+4} + 72588\right]$

11) $\frac{1}{110}\left[57965y_{4n+5} - 2415x_{4n+6} + 231860y_{2n+3} - 9660x_{2n+4} + 660\right]$

12) $\frac{1}{2}\left[265629x_{4n+6} - 57965y_{4n+6} + 1062516x_{2n+4} - 231860y_{2n+4} + 12\right]$

13) $\frac{1}{24}\left[115y_{4n+4} - y_{4n+5} + 460y_{2n+2} - 4y_{2n+3} + 144\right]$

14) $\frac{1}{2640}\left[12649y_{4n+4} - y_{4n+6} + 50596y_{2n+2} - 4y_{2n+4} + 15840\right]$

15) $\frac{1}{24}\left[12649y_{4n+5} - 115y_{4n+6} + 50596y_{2n+3} - 460y_{2n+4} + 144\right]$

D. Each Of The Following Expressions Is A Quintic Integer

1) $\frac{1}{48}\left[100x_{n+2} - 10540x_{n+1} + 50x_{3n+4} - 5270x_{3n+3} + 10x_{5n+6} - 1054x_{5n+5}\right]$

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2) $\frac{1}{528}\left[10x_{n+3}-115930x_{n+1}+5x_{3n+5}-57965x_{3n+3}+x_{5n+7}-11593x_{5n+5}\right]$

3) $\frac{1}{2}\left[50y_{n+1}-210x_{n+1}+25y_{3n+3}-105x_{3n+3}+5y_{5n+5}-21x_{5n+5}\right]$

4) $\frac{1}{22}\left[10y_{n+2}-4830x_{n+1}+5y_{3n+4}-2415x_{3n+3}+y_{5n+6}-483x_{5n+5}\right]$

5) $\frac{1}{12098}\left[50y_{n+3}-2656290x_{n+1}+25y_{3n+5}-1328145x_{3n+3}+5y_{5n+7}-265629x_{5n+5}\right]$

6) $\frac{1}{24}\left[5270x_{n+3}-579650x_{n+2}+2635x_{3n+5}-289825x_{3n+4}+527x_{5n+7}-57965x_{5n+6}\right]$

7) $\frac{1}{110}\left[5270y_{n+1}-210x_{n+2}+2635y_{3n+3}-105x_{3n+4}+527y_{5n+5}-21x_{5n+6}\right]$

8) $\frac{1}{2}\left[5270y_{n+2}-24150x_{n+2}+2635y_{3n+4}-12075x_{3n+4}+527y_{5n+6}-2415x_{5n+6}\right]$

9) $\frac{1}{110}\left[5270y_{n+3}-3656290x_{n+2}+2635y_{3n+5}-1328145x_{3n+4}+527y_{5n+7}-265629x_{5n+6}\right]$

10) $\frac{1}{12098}\left[579650y_{n+1}-210x_{n+3}+289825y_{3n+3}-105x_{3n+5}+57965y_{5n+5}-21x_{5n+7}\right]$

11) $\frac{1}{110}\left[579650y_{n+2}-24150x_{n+3}+289825y_{3n+4}-12075x_{3n+5}+57965y_{5n+6}-2415x_{5n+7}\right]$

12) $\frac{1}{2}\left[579650y_{n+3}-2656290x_{n+3}+289825y_{3n+5}-1328145x_{3n+5}+57965y_{5n+7}-265629x_{5n+7}\right]$

13) $\frac{1}{24}\left[1150y_{n+1}-10y_{n+2}+575y_{3n+3}-5y_{3n+4}+115y_{5n+5}-y_{5n+6}\right]$

14) $\frac{1}{2640}\left[126490y_{n+1}-10y_{n+3}+63245y_{3n+3}-5y_{3n+5}+12649y_{5n+5}-y_{5n+7}\right]$

15) $\frac{1}{24}\left[126490y_{n+2}-1150y_{n+3}+63245y_{3n+4}-575y_{3n+5}+12649y_{5n+6}-115y_{5n+7}\right]$

E. Relations Among The Solutions Are Given Below

1) $4x_{n+2}-220x_{n+1}-48x_{n+3}=0$
2) $48x_{n+1}-5280x_{n+2}+48x_{n+3}=0$
3) $4x_{n+1}-220x_{n+2}+48y_{n+2}=0$
4) $220x_{n+1}-24196x_{n+2}+148y_{n+3}=0$
5) $24196x_{n+1}-4x_{n+3}+5280y_{n+1}=0$
6) $24196x_{n+2}-220x_{n+3}+48y_{n+1}=0$
7) $220x_{n+1}-220x_{n+3}+5280y_{n+2}=0$
8) $220x_{n+2}-4x_{n+3}+48y_{n+2}=0$
9) $4x_{n+1}-24196x_{n+3}+5280y_{n+3}=0$

10) $4x_{n+2} - 220x_{n+3} + 48y_{n+3} = 0$

11) $1008x_{n+1} + 220y_{n+1} - 4y_{n+2} = 0$

12) $1008x_{n+2} + 4y_{n+1} - 220y_{n+2} = 0$

13) $1008x_{n+3} + 220y_{n+1} - 24196y_{n+2} = 0$

14) $110880x_{n+1} + 24196y_{n+1} - 4y_{n+3} = 0$

15) $110880x_{n+2} + 220y_{n+1} - 220y_{n+3} = 0$

16) $110880x_{n+3} + 4y_{n+1} - 24196y_{n+3} = 0$

17) $1008x_{n+1} + 24196y_{n+2} - 220y_{n+3} = 0$

18) $1008x_{n+2} + 220y_{n+2} - 4y_{n+3} = 0$

19) $1008x_{n+3} + 4y_{n+2} - 220y_{n+3} = 0$

20) $48y_{n+1} - 5280y_{n+2} + 48y_{n+3} = 0$

IV. REMARKABLE OBSERVATIONS

A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below:

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$X^2 - 84Y^2 = 9216$	$(10x_{n+2} - 1054x_{n+1}, 115x_{n+1} - x_{n+2})$
2	$25X^2 - 21Y^2 = 27878400$	$(x_{n+3} - 11593x_{n+1}, 12649x_{n+1} - x_{n+3})$
3	$X^2 - 21Y^2 = 16$	$(5y_{n+1} - 21x_{n+1}, 5x_{n+1} - y_{n+1})$
4	$25X^2 - 21Y^2 = 48400$	$(y_{n+2} - 483x_{n+1}, 527x_{n+1} - y_{n+2})$
5	$X^2 - 21Y^2 = 585446416$	$(5y_{n+3} - 265629x_{n+1}, 57965x_{n+1} - y_{n+3})$
6	$X^2 - 21Y^2 = 2304$	$(527x_{n+3} - 57965x_{n+2}, 12649x_{n+2} - 1153x_{n+3})$
7	$X^2 - 525Y^2 = 48400$	$(527y_{n+1} - 21x_{n+2}, x_{n+2} - 23y_{n+1})$
8	$X^2 - 21Y^2 = 16$	$(527y_{n+2} - 2415x_{n+2}, 527x_{n+2} - 115y_{n+2})$
9	$X^2 - 21Y^2 = 48400$	$(527y_{n+3} - 265629x_{n+2}, 57965x_{n+2} - 115y_{n+3})$
10	$X^2 - 21Y^2 = 585446416$	$(57965y_{n+1} - 21x_{n+3}, 5x_{n+3} - 12649y_{n+1})$
11	$X^2 - 21Y^2 = 48400$	$(57965y_{n+2} - 2415x_{n+3}, 527x_{n+3} - 12649y_{n+2})$
12	$X^2 - 21Y^2 = 16$	$(57965y_{n+3} - 265629x_{n+3}, 57965x_{n+3} - 12649y_{n+}$
13	$21X^2 - Y^2 = 48384$	$(115y_{n+1} - y_{n+2}, 5y_{n+2} - 527y_{n+1})$
14	$21X^2 - Y^2 = 585446400$	$(12649y_{n+1} - y_{n+3}, 5y_{n+3} - 57965y_{n+1})$
15	$21X^2 - Y^2 = 48384$	$(12649y_{n+1} - 115y_{n+3}, 527y_{n+3} - 57965y_{n+2})$

B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below:

Table:3 Parabola

S.No	Parabola	(X,Y)
1	$12X-21Y^2=1152$	$(10x_{2n+3}-1054x_{2n+2}, 115x_{n+1}-x_{n+2})$
2	$4400X-7Y^2=4646400$	$(x_{2n+4}-11593x_{2n+2}, 12649x_{n+1}-x_{n+3})$
3	$2X-21Y^2=8$	$(5y_{2n+2}-21x_{2n+2}, 5x_{n+1}-y_{n+1})$
4	$550X-21Y^2=24200$	$(y_{2n+3}-483x_{2n+2}, 527x_{n+1}-y_{n+2})$
5	$12098X-21Y^2=292723208$	$(5y_{2n+4}-262629x_{2n+2}, 57965x_{n+1}-y_{n+3})$
6	$24X-21Y^2=1152$	$(527x_{2n+4}-57965x_{2n+3}, 12649x_{n+2}-115x_{n+3})$
7	$484X-2310Y^2=106480$	$(527y_{2n+2}-21x_{2n+3}, x_{n+2}-23y_{n+1})$
8	$2X-21Y^2=8$	$(527y_{2n+3}-2415x_{2n+3}, 527x_{n+2}-115y_{n+2})$
9	$110X-21Y^2=24200$	$(527y_{2n+4}-265629x_{2n+3}, 57965x_{n+2}-115y_{n+3})$
10	$12098X-21Y^2=292723208$	$(57965y_{2n+2}-21x_{2n+4}, 5x_{n+3}-12649y_{n+1})$
11	$110X-21Y^2=24200$	$(57965y_{2n+3}-2415x_{2n+4}, 527x_{n+3}-12649y_{n+2})$
12	$2X-21Y^2=8$	$(57965y_{2n+4}-265629x_{2n+4}, 57965x_{n+3}-12649y_{n+3})$
13	$504X-Y^2=24192$	$(115y_{2n+2}-y_{2n+3}, 527y_{n+1}-5y_{n+2})$
14	$55440X-Y^2=292723200$	$(12649y_{2n+2}-y_{2n+4}, 57965y_{n+1}-5y_{n+3})$
15	$504X-Y^2=24192$	$(12649y_{2n+3}-115y_{2n+4}, 57965y_{n+2}-527y_{n+3})$

1) Some special cases of the solutions are given below

a) $P_y^{10}(t_{3,x+1})^2=189P_x^6(t_{3,y})^2+4(t_{3,y})^2(t_{3,x+1})^2$

b) $9P_y^6(t_{3,x})^2=21P_x^{10}(t_{3,y+1})^2+4(t_{3,x})^2(t_{3,y+1})^2$

c) $P_y^{10}(t_{3,2x-2})^2=21(6P_{x-1}^4)^2(t_{3,y})^2+4(t_{3,y})^2(t_{3,2x-2})^2$

d) $36P_{y-1}^8(t_{3,x})^2=21P_x^{10}(t_{3,2y-2})^2+4(t_{3,x})^2(t_{3,2y-2})^2$

e) $9P_y^6(t_{3,2x-2})^2=21(36P_{x-1}^8)(t_{3,y+1})^2+4(t_{3,2x-2})^2(t_{3,y+1})^2$

f) $(6P_{y-1}^4)^2(t_{3,x+1})^2=21(3P_x^3)^2(t_{3,2y-2})^2+8(t_{3,x+1})^2(t_{3,2y-2})^2$

V. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2=21x^2+4$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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