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# On the Positive Pell Equation $y^2 = 21x^2 + 4$

$$y^2 = 21x^2 + 4$$

Dr. A. Kavitha<sup>1</sup>, K. Megala<sup>2</sup>

<sup>1</sup>Assistant Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India.

<sup>2</sup>M. Phil Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India.

**Abstract:** The binary quadratic Diophantine equation represented by the positive Pellian  $y^2 = 21x^2 + 4$  is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions we have obtained the solutions of other choices of hyperbolas and parabolas.

**Keywords:** Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

## I. INTRODUCTION

The binary quadratic equations of the form  $y^2 = Dx^2 + 1$  where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4]. For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by  $y^2 = 21x^2 + 4$  is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

## II. NOTATIONS

$t_{m,n} = n \left( 1 + \frac{(n-1)(m-2)}{2} \right)$  polygonal number of rank n with size m

$p^m = \frac{1}{6} n(n+1)((m-2)n+5-m)$  Pyramidal number of rank n with size m

## III. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 21x^2 + 4 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 1, y_0 = 5 \quad D = 21$$

consider the Pellian equation is

$$y^2 = 21x^2 + 1 \tag{2}$$

The initial solution of Pellian equation is

$$\tilde{x}_0 = 12, \tilde{y}_0 = 55,$$

The general solution  $(\tilde{x}_n, \tilde{y}_n)$  of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{21}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (55 + 12\sqrt{21})^{n+1} + (55 - 12\sqrt{21})^{n+1}$$

$$g_n = (55 + 12\sqrt{21})^{n+1} - (55 - 12\sqrt{21})^{n+1}$$

Applying Brahmagupta lemma between  $(x_0, y_0)$  and  $(\tilde{x}_n, \tilde{y}_n)$  the other integer solution of (1) are given by,

$$x_{n+1} = \frac{1}{2}f_n + \frac{5}{2\sqrt{21}}g_n$$

$$y_{n+1} = \frac{5}{2}f_n + \frac{\sqrt{21}}{2}g_n$$

The recurrence relation satisfied by the solution  $x$  and  $y$  are given by,

$$x_{n+3} - 110x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 110y_{n+2} + y_{n+1} = 0$$

Some numerical examples of  $x_n$  and  $y_n$  satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	$x_n$	$y_n$
0	1	5
1	115	527
2	12649	57965
3	1391275	6375623
4	153027601	701260565

From the above table, we observe some interesting relations among the solutions which are presented below,

Both  $x_n$  and  $y_n$  values are odd.

A. Each Of The Following Expression Is A Nasty Number

1)  $\frac{1}{4}[5x_{2n+3} - 527x_{2n+2} + 48]$

2)  $\frac{1}{176}[2x_{2n+4} - 23186x_{2n+2} + 2112]$

3)  $\frac{1}{2}[30y_{2n+2} - 126x_{2n+2} + 24]$

4)  $\frac{1}{11}[3y_{2n+3} - 2949x_{2n+2} + 132]$

5)  $\frac{1}{6049}[15y_{2n+4} - 796887x_{2n+2} + 72588]$

6)  $\frac{1}{4}[527x_{2n+4} - 57965x_{2n+3} + 48]$

7)  $\frac{1}{55}[1581y_{2n+2} - 63x_{2n+3} + 660]$

8)  $\frac{1}{2}[3162y_{2n+3} - 14490x_{2n+3} + 12]$

9)  $\frac{1}{55}[1581y_{2n+4} - 796887x_{2n+3} + 660]$

$$10) \frac{1}{6049} [173895y_{2n+2} - 63x_{2n+4} + 72588]$$

$$11) \frac{1}{55} [173895y_{2n+3} - 7245x_{2n+4} + 660]$$

$$12) \frac{1}{2} [347790y_{2n+4} - 1593774x_{2n+4} + 24]$$

$$13) \frac{1}{24} [690y_{2n+2} - 6y_{2n+3} + 288]$$

$$14) \frac{1}{1320} [37947y_{2n+2} - 3y_{2n+4} + 15840]$$

$$15) \frac{1}{12} [12649y_{2n+3} - 115y_{2n+4} + 48]$$

*B. Each Of The Following Expressions Is A Cubical Integer*

$$1) \frac{1}{48} [10x_{3n+4} - 1054x_{3n+3} + 30x_{n+2} - 3162x_{n+1}]$$

$$2) \frac{1}{528} [x_{3n+5} - 11593x_{3n+3} + 3x_{n+3} - 34779x_{n+1}]$$

$$3) \frac{1}{2} [5y_{3n+3} - 21x_{3n+3} + 15y_{n+1} - 63x_{n+1}]$$

$$4) \frac{1}{22} [y_{3n+4} - 483x_{3n+3} + 3y_{n+2} - 1449x_{n+1}]$$

$$5) \frac{1}{12098} [5y_{3n+5} - 265629x_{3n+3} + 15y_{n+3} - 796887x_{n+1}]$$

$$6) \frac{1}{24} [527x_{3n+5} - 57965x_{3n+4} + 1581x_{n+3} - 173895x_{n+2}]$$

$$7) \frac{1}{110} [527y_{3n+3} - 21x_{3n+4} + 1581y_{n+1} - 63x_{n+2}]$$

$$8) \frac{1}{2} [527y_{3n+4} - 2415x_{3n+4} + 1581y_{n+2} - 7245x_{n+2}]$$

$$9) \frac{1}{110} [527y_{3n+5} - 265629x_{3n+4} + 1581y_{n+3} - 796887x_{n+2}]$$

$$10) \frac{1}{12098} [57965y_{3n+3} - 21x_{3n+5} + 173895y_{n+1} - 63x_{n+3}]$$

$$11) \frac{1}{110} [57965y_{3n+4} - 2415x_{3n+5} + 173895y_{n+2} - 7245x_{n+3}]$$

$$12) \frac{1}{2} [57965y_{3n+5} - 265629x_{3n+5} + 173895y_{n+3} - 796887x_{n+3}]$$

$$13) \frac{1}{24} [115y_{3n+3} - y_{3n+4} + 345y_{n+1} - 3y_{n+2}]$$

$$14) \frac{1}{2640} [12649y_{3n+3} - y_{3n+5} + 37947y_{n+1} - 3y_{n+3}]$$

$$15) \frac{1}{24} [12649y_{3n+4} - 115y_{3n+5} + 37947y_{n+2} - 345y_{n+3}]$$

C. Each Of The Following Expressions Is A Bi-Quadratic Integer

$$1) \frac{1}{48} [10x_{4n+5} - 1054x_{4n+4} + 40x_{2n+3} - 4216x_{2n+2} + 288]$$

$$2) \frac{1}{528} [x_{4n+6} - 11593x_{4n+4} + 4x_{2n+4} - 46372x_{2n+2} + 3168]$$

$$3) \frac{1}{2} [5y_{4n+4} - 21x_{4n+4} + 20y_{2n+2} - 84x_{2n+2} + 12]$$

$$4) \frac{1}{22} [y_{4n+5} - 483x_{4n+4} + 4y_{2n+3} - 1932x_{2n+2} + 132]$$

$$5) \frac{1}{12098} [5y_{4n+6} - 265629x_{4n+4} + 20y_{2n+4} - 1062516x_{2n+2} + 72588]$$

$$6) \frac{1}{24} [527x_{4n+6} - 57965x_{4n+5} + 2108x_{2n+4} - 231860x_{2n+3} + 144]$$

$$7) \frac{1}{110} [527y_{4n+4} - 21x_{4n+5} + 2108y_{2n+2} - 84x_{2n+3} + 660]$$

$$8) \frac{1}{2} [527y_{4n+5} - 2415x_{4n+5} + 2108y_{2n+3} - 9660x_{2n+3} + 12]$$

$$9) \frac{1}{110} [527y_{4n+6} - 265629x_{4n+5} + 2108y_{2n+4} - 1062516x_{2n+3} + 660]$$

$$10) \frac{1}{12098} [57965y_{4n+4} - 21x_{4n+6} + 231860y_{2n+2} - 84x_{2n+4} + 72588]$$

$$11) \frac{1}{110} [57965y_{4n+5} - 2415x_{4n+6} + 231860y_{2n+3} - 9660x_{2n+4} + 660]$$

$$12) \frac{1}{2} [265629x_{4n+6} - 57965y_{4n+6} + 1062516x_{2n+4} - 231860y_{2n+4} + 12]$$

$$13) \frac{1}{24} [115y_{4n+4} - y_{4n+5} + 460y_{2n+2} - 4y_{2n+3} + 144]$$

$$14) \frac{1}{2640} [12649y_{4n+4} - y_{4n+6} + 50596y_{2n+2} - 4y_{2n+4} + 15840]$$

$$15) \frac{1}{24} [12649y_{4n+5} - 115y_{4n+6} + 50596y_{2n+3} - 460y_{2n+4} + 144]$$

D. Each Of The Following Expressions Is A Quintic Integer

$$1) \frac{1}{48} [100x_{n+2} - 10540x_{n+1} + 50x_{3n+4} - 5270x_{3n+3} + 10x_{5n+6} - 1054x_{5n+5}]$$

- 2)  $\frac{1}{528} [10x_{n+3} - 115930x_{n+1} + 5x_{3n+5} - 57965x_{3n+3} + x_{5n+7} - 11593x_{5n+5}]$
- 3)  $\frac{1}{2} [50y_{n+1} - 210x_{n+1} + 25y_{3n+3} - 105x_{3n+3} + 5y_{5n+5} - 21x_{5n+5}]$
- 4)  $\frac{1}{22} [10y_{n+2} - 4830x_{n+1} + 5y_{3n+4} - 2415x_{3n+3} + y_{5n+6} - 483x_{5n+5}]$
- 5)  $\frac{1}{12098} [50y_{n+3} - 2656290x_{n+1} + 25y_{3n+5} - 1328145x_{3n+3} + 5y_{5n+7} - 265629x_{5n+5}]$
- 6)  $\frac{1}{24} [5270x_{n+3} - 579650x_{n+2} + 2635x_{3n+5} - 289825x_{3n+4} + 527x_{5n+7} - 57965x_{5n+6}]$
- 7)  $\frac{1}{110} [5270y_{n+1} - 210x_{n+2} + 2635y_{3n+3} - 105x_{3n+4} + 527y_{5n+5} - 21x_{5n+6}]$
- 8)  $\frac{1}{2} [5270y_{n+2} - 24150x_{n+2} + 2635y_{3n+4} - 12075x_{3n+4} + 527y_{5n+6} - 2415x_{5n+6}]$
- 9)  $\frac{1}{110} [5270y_{n+3} - 3656290x_{n+2} + 2635y_{3n+5} - 1328145x_{3n+4} + 527y_{5n+7} - 265629x_{5n+6}]$
- 10)  $\frac{1}{12098} [579650y_{n+1} - 210x_{n+3} + 289825y_{3n+3} - 105x_{3n+5} + 57965y_{5n+5} - 21x_{5n+7}]$
- 11)  $\frac{1}{110} [579650y_{n+2} - 24150x_{n+3} + 289825y_{3n+4} - 12075x_{3n+5} + 57965y_{5n+6} - 2415x_{5n+7}]$
- 12)  $\frac{1}{2} [579650y_{n+3} - 2656290x_{n+3} + 289825y_{3n+5} - 1328145x_{3n+5} + 57965y_{5n+7} - 265629x_{5n+7}]$
- 13)  $\frac{1}{24} [1150y_{n+1} - 10y_{n+2} + 575y_{3n+3} - 5y_{3n+4} + 115y_{5n+5} - y_{5n+6}]$
- 14)  $\frac{1}{2640} [126490y_{n+1} - 10y_{n+3} + 63245y_{3n+3} - 5y_{3n+5} + 12649y_{5n+5} - y_{5n+7}]$
- 15)  $\frac{1}{24} [126490y_{n+2} - 1150y_{n+3} + 63245y_{3n+4} - 575y_{3n+5} + 12649y_{5n+6} - 115y_{5n+7}]$

*E. Relations Among The Solutions Are Given Below*

- 1)  $4x_{n+2} - 220x_{n+1} - 48x_{n+3} = 0$
- 2)  $48x_{n+1} - 5280x_{n+2} + 48x_{n+3} = 0$
- 3)  $4x_{n+1} - 220x_{n+2} + 48y_{n+2} = 0$
- 4)  $220x_{n+1} - 24196x_{n+2} + 148y_{n+3} = 0$
- 5)  $24196x_{n+1} - 4x_{n+3} + 5280y_{n+1} = 0$
- 6)  $24196x_{n+2} - 220x_{n+3} + 48y_{n+1} = 0$
- 7)  $220x_{n+1} - 220x_{n+3} + 5280y_{n+2} = 0$
- 8)  $220x_{n+2} - 4x_{n+3} + 48y_{n+2} = 0$
- 9)  $4x_{n+1} - 24196x_{n+3} + 5280y_{n+3} = 0$

- 10)  $4x_{n+2} - 220x_{n+3} + 48y_{n+3} = 0$
- 11)  $1008x_{n+1} + 220y_{n+1} - 4y_{n+2} = 0$
- 12)  $1008x_{n+2} + 4y_{n+1} - 220y_{n+2} = 0$
- 13)  $1008x_{n+3} + 220y_{n+1} - 24196y_{n+2} = 0$
- 14)  $110880x_{n+1} + 24196y_{n+1} - 4y_{n+3} = 0$
- 15)  $110880x_{n+2} + 220y_{n+1} - 220y_{n+3} = 0$
- 16)  $110880x_{n+3} + 4y_{n+1} - 24196y_{n+3} = 0$
- 17)  $1008x_{n+1} + 24196y_{n+2} - 220y_{n+3} = 0$
- 18)  $1008x_{n+2} + 220y_{n+2} - 4y_{n+3} = 0$
- 19)  $1008x_{n+3} + 4y_{n+2} - 220y_{n+3} = 0$
- 20)  $48y_{n+1} - 5280y_{n+2} + 48y_{n+3} = 0$

#### IV. REMARKABLE OBSERVATIONS

- A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below:

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$X^2 - 84Y^2 = 9216$	$(10x_{n+2} - 1054x_{n+1}, 115x_{n+1} - x_{n+2})$
2	$25X^2 - 21Y^2 = 27878400$	$(x_{n+3} - 11593x_{n+1}, 12649x_{n+1} - x_{n+3})$
3	$X^2 - 21Y^2 = 16$	$(5y_{n+1} - 21x_{n+1}, 5x_{n+1} - y_{n+1})$
4	$25X^2 - 21Y^2 = 48400$	$(y_{n+2} - 483x_{n+1}, 527x_{n+1} - y_{n+2})$
5	$X^2 - 21Y^2 = 585446416$	$(5y_{n+3} - 265629x_{n+1}, 57965x_{n+1} - y_{n+3})$
6	$X^2 - 21Y^2 = 2304$	$(527x_{n+3} - 57965x_{n+2}, 12649x_{n+2} - 1153x_{n+3})$
7	$X^2 - 525Y^2 = 48400$	$(527y_{n+1} - 21x_{n+2}, x_{n+2} - 23y_{n+1})$
8	$X^2 - 21Y^2 = 16$	$(527y_{n+2} - 2415x_{n+2}, 527x_{n+2} - 115y_{n+2})$
9	$X^2 - 21Y^2 = 48400$	$(527y_{n+3} - 265629x_{n+2}, 57965x_{n+2} - 115y_{n+3})$
10	$X^2 - 21Y^2 = 585446416$	$(57965y_{n+1} - 21x_{n+3}, 5x_{n+3} - 12649y_{n+1})$
11	$X^2 - 21Y^2 = 48400$	$(57965y_{n+2} - 2415x_{n+3}, 527x_{n+3} - 12649y_{n+2})$
12	$X^2 - 21Y^2 = 16$	$(57965y_{n+3} - 265629x_{n+3}, 57965x_{n+3} - 12649y_{n+3})$
13	$21X^2 - Y^2 = 48384$	$(115y_{n+1} - y_{n+2}, 5y_{n+2} - 527y_{n+1})$
14	$21X^2 - Y^2 = 585446400$	$(12649y_{n+1} - y_{n+3}, 5y_{n+3} - 57965y_{n+1})$
15	$21X^2 - Y^2 = 48384$	$(12649y_{n+1} - 115y_{n+3}, 527y_{n+3} - 57965y_{n+2})$

B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below:

Table:3 Parabola

S.No	Parabola	(X,Y)
1	$12X - 21Y^2 = 1152$	$(10x_{2n+3} - 1054x_{2n+2}, 115x_{n+1} - x_{n+2})$
2	$4400X - 7Y^2 = 4646400$	$(x_{2n+4} - 11593x_{2n+2}, 12649x_{n+1} - x_{n+3})$
3	$2X - 21Y^2 = 8$	$(5y_{2n+2} - 21x_{2n+2}, 5x_{n+1} - y_{n+1})$
4	$550X - 21Y^2 = 24200$	$(y_{2n+3} - 483x_{2n+2}, 527x_{n+1} - y_{n+2})$
5	$12098X - 21Y^2 = 292723208$	$(5y_{2n+4} - 262629x_{2n+2}, 57965x_{n+1} - y_{n+3})$
6	$24X - 21Y^2 = 1152$	$(527x_{2n+4} - 57965x_{2n+3}, 12649x_{n+2} - 115x_{n+3})$
7	$484X - 2310Y^2 = 106480$	$(527y_{2n+2} - 21x_{2n+3}, x_{n+2} - 23y_{n+1})$
8	$2X - 21Y^2 = 8$	$(527y_{2n+3} - 2415x_{2n+3}, 527x_{n+2} - 115y_{n+2})$
9	$110X - 21Y^2 = 24200$	$(527y_{2n+4} - 265629x_{2n+3}, 57965x_{n+2} - 115y_{n+3})$
10	$12098X - 21Y^2 = 292723208$	$(57965y_{2n+2} - 21x_{2n+4}, 5x_{n+3} - 12649y_{n+1})$
11	$110X - 21Y^2 = 24200$	$(57965y_{2n+3} - 2415x_{2n+4}, 527x_{n+3} - 12649y_{n+2})$
12	$2X - 21Y^2 = 8$	$(57965y_{2n+4} - 265629x_{2n+4}, 57965x_{n+3} - 12649y_{n+3})$
13	$504X - Y^2 = 24192$	$(115y_{2n+2} - y_{2n+3}, 527y_{n+1} - 5y_{n+2})$
14	$55440X - Y^2 = 292723200$	$(12649y_{2n+2} - y_{2n+4}, 57965y_{n+1} - 5y_{n+3})$
15	$504X - Y^2 = 24192$	$(12649y_{2n+3} - 115y_{2n+4}, 57965y_{n+2} - 527y_{n+3})$

1) Some special cases of the solutions are given below

a)  $P_y^{10}(t_{3,x+1})^2 = 189P_x^6(t_{3,y})^2 + 4(t_{3,y})^2(t_{3,x+1})^2$

b)  $9P_y^6(t_{3,x})^2 = 21P_x^{10}(t_{3,y+1})^2 + 4(t_{3,x})^2(t_{3,y+1})^2$

c)  $P_y^{10}(t_{3,2x-2})^2 = 21(6P_{x-1}^4)^2(t_{3,y})^2 + 4(t_{3,y})^2(t_{3,2x-2})^2$

d)  $36P_{y-1}^8(t_{3,x})^2 = 21P_x^{10}(t_{3,2y-2})^2 + 4(t_{3,x})^2(t_{3,2y-2})^2$

e)  $9P_y^6(t_{3,2x-2})^2 = 21(36P_{x-1}^8)(t_{3,y+1})^2 + 4(t_{3,2x-2})^2(t_{3,y+1})^2$

f)  $(6P_{y-1}^4)^2(t_{3,x+1})^2 = 21(3P_x^3)^2(t_{3,2y-2})^2 + 8(t_{3,x+1})^2(t_{3,2y-2})^2$

### V. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations  $y^2 = 21x^2 + 4$  As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.



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