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On the Positive Pell Equation $y^2 = 17x^2 + 8$

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Abstract: The binary quadratic Diophantine equation $y^2 = 17x^2 + 8$ is analyzed for its non-zero distinct integral solutions. A few interesting relations among the solutions are given. Further, employing the solutions have obtained solutions of other choices of hyperbolas and parabolas.

Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4]. For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 17x^2 + 8$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented

II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 17x^2 + 8 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 1, y_0 = 5 \quad D = 17$$

consider the Pellian equation is

$$y^2 = 17x^2 + 1 \tag{2}$$

The initial solution of Pellian equation is

$$\tilde{x}_0 = 8, \tilde{y}_0 = 33,$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{17}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (33 + 8\sqrt{17})^{n+1} + (33 - 8\sqrt{17})^{n+1}$$

$$g_n = (33 + 8\sqrt{17})^{n+1} - (33 - 8\sqrt{17})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{17}} g_n$$

$$y_{n+1} = \frac{5}{2} f_n + \frac{17}{2\sqrt{17}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 66x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 66y_{n+2} + y_{n+1} = 0$$

Where $n=0,1,2,3,\dots$

Some numerical examples of x_n and y_n satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	1	5
1	73	301
2	4817	19861
3	317849	1310525
4	20973217	86474789

From the above table, we observe some interesting relations among the solutions which are presented below.

A. Both x_n and y_n values are odd.

B. Each Of The Following Expression Is A Nasty Number

$$1) \frac{3}{2} [8 + 5y_{2n+2} - 17x_{2n+2}]$$

$$2) \frac{3}{16} [64 + 5x_{2n+3} - 301x_{2n+2}]$$

$$3) \frac{1}{352} [4224 + 5x_{2n+4} - 19861x_{2n+2}]$$

$$4) \frac{1}{22} [264 + 5y_{2n+3} - 1241x_{2n+2}]$$

$$5) \frac{3}{4354} [17416 + 5y_{2n+4} - 81889x_{2n+2}]$$

$$6) \frac{1}{22} [264 + 301y_{2n+2} - 17x_{2n+3}]$$

$$7) \frac{3}{4354} [17416 + 19861y_{2n+2} - 17x_{2n+4}]$$

$$8) \frac{3}{272} [1088 + 1241y_{2n+2} - 17y_{2n+3}]$$

$$9) \frac{1}{5984} [71808 + 81889y_{2n+2} - 17y_{2n+4}]$$

$$10) \frac{3}{2} [8 + 301y_{2n+3} - 1241x_{2n+3}]$$

$$11) \frac{3}{16} [64 + 301x_{2n+4} - 19861x_{2n+3}]$$

$$12) \frac{1}{22} [264 + 301y_{2n+4} - 81889x_{2n+3}]$$

$$13) \frac{1}{22} [264 + 19861y_{2n+3} - 1241x_{2n+4}]$$

$$14) \frac{3}{2} [8 + 19861y_{2n+4} - 81889x_{2n+4}]$$

$$15) \frac{3}{272} [1088 + 81889y_{2n+3} - 1241y_{2n+4}]$$

C. Each Of The Following Expressions Is A Cubical Integer

$$1) \frac{1}{4} [5y_{3n+3} - 17x_{3n+3} + 15y_{n+1} - 51x_{n+1}]$$

$$2) \frac{1}{32} [5x_{3n+4} - 301x_{3n+3} + 15x_{n+2} - 903x_{n+1}]$$

$$3) \frac{1}{2112} [5x_{3n+5} - 19861x_{3n+3} + 15x_{n+3} - 59583x_{n+1}]$$

$$4) \frac{1}{132} [5y_{3n+4} - 1241x_{3n+3} + 15y_{n+2} - 3723x_{n+1}]$$

$$5) \frac{1}{8708} [5y_{3n+5} - 81889x_{3n+3} + 15y_{n+3} - 245667x_{n+1}]$$

$$6) \frac{1}{132} [301y_{3n+3} - 17x_{3n+4} + 903y_{n+1} - 51x_{n+2}]$$

$$7) \frac{1}{8708} [19861y_{3n+3} - 17x_{3n+5} + 59583y_{n+1} - 51x_{n+3}]$$

$$8) \frac{1}{544} [1241y_{3n+3} - 17y_{3n+4} + 3723y_{n+1} - 51y_{n+2}]$$

$$9) \frac{1}{35904} [81889y_{3n+3} - 17y_{3n+5} + 245667y_{n+1} - 51y_{n+3}]$$

$$10) \frac{1}{32} [301x_{3n+5} - 19861x_{3n+4} + 903x_{n+3} - 59583x_{n+2}]$$

$$11) \frac{1}{4} [301y_{3n+4} - 1241x_{3n+4} + 903y_{n+2} - 3723x_{n+2}]$$

$$12) \frac{1}{132} [301y_{3n+5} - 81889x_{3n+4} + 903y_{n+3} - 245667x_{n+2}]$$

$$13) \frac{1}{132} [19861y_{3n+4} - 1241x_{3n+5} + 59583y_{n+2} - 3723x_{n+3}]$$

$$14) \frac{1}{4} [19861y_{3n+5} - 81889x_{3n+5} + 59583y_{n+3} - 245667x_{n+3}]$$

$$15) \frac{1}{544} [81889y_{3n+4} - 1241y_{3n+5} + 245667y_{n+2} - 3723y_{n+3}]$$

D. Each Of The Following Expressions Is A Biquadratic Integer

$$1) \frac{1}{4} [5y_{4n+4} - 17x_{4n+4} + 20y_{2n+2} - 68x_{2n+2} + 24]$$

$$2) \frac{1}{32} [5x_{4n+5} - 301x_{4n+4} + 20x_{2n+3} - 1204x_{2n+2} + 192]$$

$$3) \frac{1}{2112} [5x_{4n+6} - 19861x_{4n+4} + 20x_{2n+4} - 79444x_{2n+2} + 12672]$$

$$4) \frac{1}{132} [5y_{4n+5} - 1241x_{4n+4} + 20y_{2n+3} - 4964x_{2n+2} + 792]$$

$$5) \frac{1}{8708} [5y_{4n+6} - 81889x_{4n+4} + 20y_{2n+4} - 327556x_{2n+2} + 52248]$$

$$6) \frac{1}{132} [301y_{4n+4} - 17x_{4n+5} + 1204y_{2n+2} - 68x_{2n+3} + 792]$$

$$7) \frac{1}{8708} [19861y_{4n+4} - 17x_{4n+6} + 79444y_{2n+2} - 68x_{2n+4} + 52248]$$

$$8) \frac{1}{544} [1241y_{4n+4} - 17y_{4n+5} + 4964y_{2n+2} - 68y_{2n+3} + 3264]$$

$$9) \frac{1}{35904} [81889y_{4n+4} - 17y_{4n+6} + 327556y_{2n+2} - 68y_{2n+4} + 215424]$$

$$10) \frac{1}{32} [301x_{4n+6} - 19861x_{4n+5} + 1204x_{2n+4} - 79444x_{2n+3} + 192]$$

$$11) \frac{1}{4} [301y_{4n+5} - 1241x_{4n+5} + 1204y_{2n+3} - 4964x_{2n+3} + 24]$$

$$12) \frac{1}{132} [301y_{4n+6} - 81889x_{4n+5} + 1204y_{2n+4} - 327556x_{2n+3} + 792]$$

$$13) \frac{1}{132} [19861y_{4n+5} - 1241x_{4n+6} + 79444y_{2n+3} - 4964x_{2n+4} + 792]$$

$$14) \frac{1}{4} [19861y_{4n+6} - 81889x_{4n+6} + 79444y_{2n+4} - 327556x_{2n+4} + 24]$$

$$15) \frac{1}{544} [81889y_{4n+5} - 1241y_{4n+6} + 327556y_{2n+3} - 4964y_{2n+4} + 3264]$$

E. Each Of The Following Expression Is A Quintic Integer

$$1) \frac{1}{4} [5y_{5n+5} - 17x_{5n+5} + 25y_{3n+3} - 85x_{3n+3} + 50y_{n+1} - 170x_{n+1}]$$

$$2) \frac{1}{32} [5x_{5n+6} - 301x_{5n+5} + 25x_{3n+4} - 1505x_{3n+3} + 50x_{n+2} - 3010x_{n+1}]$$

$$3) \frac{1}{2112} [5x_{5n+7} - 19861x_{5n+5} + 25x_{3n+5} - 99305x_{3n+3} + 50x_{n+3} - 198610x_{n+1}]$$

$$4) \frac{1}{132} [5y_{5n+6} - 1241x_{5n+5} + 25y_{3n+4} - 6205x_{3n+3} + 50y_{n+2} - 12410x_{n+1}]$$

$$5) \frac{1}{8708} [5y_{5n+7} - 81889x_{5n+5} + 25y_{3n+5} - 409445x_{3n+3} + 50y_{n+3} - 818890x_{n+1}]$$

$$6) \frac{1}{132} [301y_{5n+5} - 17x_{5n+6} + 1505y_{3n+3} - 85x_{3n+4} + 3010y_{n+1} - 170x_{n+2}]$$

- 7) $\frac{1}{544} [1241y_{5n+5} - 17y_{5n+6} + 6205y_{3n+3} - 85y_{3n+4} + 12410y_{n+1} - 170y_{n+2}]$
- 8) $\frac{1}{2112} [4817y_{5n+5} - y_{5n+7} + 24085y_{3n+3} - 5y_{3n+5} + 48170y_{n+1} - 10y_{n+3}]$
- 9) $\frac{1}{32} [301x_{5n+7} - 19861x_{5n+6} + 1505x_{3n+5} - 99305x_{3n+4} + 3010x_{n+3} - 198610x_{n+2}]$
- 10) $\frac{1}{4} [301y_{5n+6} - 1241x_{5n+6} + 1505y_{3n+4} - 6205x_{3n+4} + 3010y_{n+2} - 12410x_{n+2}]$
- 11) $\frac{1}{132} [301y_{5n+7} - 81889x_{5n+6} + 1505y_{3n+5} - 409445x_{3n+4} + 3010y_{n+3} - 818890x_{n+2}]$
- 12) $\frac{1}{132} [19861y_{5n+6} - 1241x_{5n+7} + 99305y_{3n+4} - 6205x_{3n+5} + 198610y_{n+2} - 12410x_{n+3}]$
- 13) $\frac{1}{4} [19861y_{5n+7} - 81889x_{5n+7} + 99305y_{3n+5} - 409445x_{3n+5} + 198610y_{n+3} - 818890x_{n+3}]$
- 14) $\frac{1}{544} [81889y_{5n+6} - 1241y_{5n+7} + 409445y_{3n+4} - 6205y_{3n+5} + 818890y_{n+2} - 12410y_{n+3}]$

F. Relations Among The Solutions Are Given Below

- 1) $x_{n+2} = 8y_{n+1} + 33x_{n+1}$
- 2) $x_{n+3} = 528y_{n+1} + 2177x_{n+1}$
- 3) $y_{n+2} = 33y_{n+1} + 136x_{n+1}$
- 4) $y_{n+3} = 2177y_{n+1} + 8976x_{n+1}$
- 5) $x_{n+3} = 66x_{n+2} - x_{n+1}$
- 6) $8y_{n+2} = 33x_{n+2} - x_{n+1}$
- 7) $8y_{n+3} = 2177x_{n+2} - 33x_{n+1}$
- 8) $16y_{n+2} = x_{n+3} - x_{n+1}$
- 9) $528y_{n+3} = 2177x_{n+3} - x_{n+1}$
- 10) $33y_{n+3} = 2177y_{n+2} + 136x_{n+1}$
- 11) $33x_{n+3} = 8y_{n+1} + 2177x_{n+2}$
- 12) $33y_{n+2} = y_{n+1} + 136x_{n+2}$
- 13) $y_{n+3} = y_{n+1} + 272x_{n+2}$
- 14) $2177y_{n+2} = 33y_{n+1} + 136x_{n+3}$
- 15) $2177y_{n+3} = y_{n+1} + 8976x_{n+3}$
- 16) $y_{n+3} = 66y_{n+2} - y_{n+1}$
- 17) $8y_{n+2} = x_{n+3} - 33x_{n+2}$
- 18) $8y_{n+3} = 33x_{n+3} - x_{n+2}$

19) $y_{n+3} = 33y_{n+2} + 136x_{n+2}$

20) $33y_{n+3} = y_{n+2} + 136x_{n+3}$

III. REMARKABLE OBSERVATIONS

A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below:

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 17X^2 = 64$	$(5x_{n+1} - y_{n+1}, 5y_{n+1} - 17x_{n+1})$
2	$Y^2 - 17X^2 = 4096$	$(73x_{n+1} - x_{n+2}, 5x_{n+2} - 301x_{n+1})$
3	$Y^2 - 17X^2 = 17842176$	$(4817x_{n+1} - x_{n+3}, 5x_{n+3} - 19861x_{n+1})$
4	$Y^2 - 17X^2 = 69696$	$(301x_{n+1} - y_{n+2}, 5y_{n+2} - 1241x_{n+1})$
5	$Y^2 - 17X^2 = 303317056$	$(19861x_{n+1} - y_{n+3}, 5y_{n+3} - 81889x_{n+1})$
6	$Y^2 - 17X^2 = 69696$	$(5x_{n+2} - 73y_{n+1}, 301y_{n+1} - 17x_{n+2})$
7	$Y^2 - 17X^2 = 303317056$	$(5x_{n+3} - 4817y_{n+1}, 19861y_{n+1} - 17x_{n+3})$
8	$Y^2 - 17X^2 = 1183744$	$(5y_{n+2} - 301y_{n+1}, 1241y_{n+1} - 17y_{n+2})$
9	$Y^2 - 17X^2 = 5156388864$	$(5y_{n+3} - 19861y_{n+1}, 81889y_{n+1} - 17y_{n+3})$
10	$Y^2 - 17X^2 = 4096$	$(4817x_{n+2} - 73x_{n+3}, 301x_{n+3} - 19861x_{n+2})$
11	$Y^2 - 17X^2 = 64$	$(301x_{n+2} - 73y_{n+2}, 301y_{n+2} - 1241x_{n+2})$
12	$Y^2 - 17X^2 = 69696$	$(19861x_{n+2} - 73y_{n+3}, 301y_{n+3} - 81889x_{n+2})$
13	$Y^2 - 17X^2 = 69696$	$(301x_{n+3} - 4817y_{n+2}, 19861y_{n+2} - 1241x_{n+3})$
14	$Y^2 - 17X^2 = 64$	$(19861x_{n+3} - 4817y_{n+3}, 19861y_{n+3} - 81889x_{n+3})$
15	$Y^2 - 17X^2 = 1183744$	$(301y_{n+3} - 19861y_{n+2}, 81889y_{n+2} - 1241y_{n+3})$

B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below:

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$4Y - 17X^2 = 64$	$(5x_{n+1} - y_{n+1}, 5y_{2n+2} - 17x_{2n+2} + 8)$
2	$32Y - 17X^2 = 4096$	$(73x_{n+1} - x_{n+2}, 5x_{2n+3} - 301x_{2n+2} + 64)$
3	$2112Y - 17X^2 = 17842176$	$(4817x_{n+1} - x_{n+3}, 5x_{2n+4} - 19861x_{2n+2} + 4224)$
4	$132Y - 17X^2 = 69696$	$(301x_{n+1} - y_{n+2}, 5y_{2n+3} - 1241x_{2n+2} + 264)$
5	$8708Y - 17X^2 = 303317056$	$(19861x_{n+1} - y_{n+3}, 5y_{2n+4} - 81889x_{2n+2} + 17416)$
6	$132Y - 17X^2 = 69696$	$(5x_{n+2} - 73y_{n+1}, 301y_{2n+2} - 17x_{2n+3} + 264)$
7	$8708Y - 17X^2 = 303317056$	$(5x_{n+3} - 4817y_{n+1}, 19861y_{2n+2} - 17x_{2n+4} + 17416)$
8	$544Y - 17X^2 = 1183744$	$(5y_{n+2} - 301y_{n+1}, 1241y_{2n+2} - 17y_{2n+3} + 1088)$
9	$35904Y - 17X^2 = 5156388864$	$(5y_{n+3} - 19861y_{n+1}, 81889y_{2n+2} - 17y_{2n+4} + 71808)$
10	$32Y - 17X^2 = 4096$	$(4817x_{n+2} - 73x_{n+3}, 301x_{2n+4} - 19861x_{2n+3} + 64)$
11	$4Y - 17X^2 = 64$	$(301x_{n+2} - 73y_{n+2}, 301y_{2n+3} - 1241x_{2n+3} + 8)$
12	$132Y - 17X^2 = 69696$	$(19861x_{n+2} - 73y_{n+3}, 301y_{2n+4} - 81889x_{2n+3} + 264)$
13	$132Y - 17X^2 = 69696$	$(301x_{n+3} - 4817y_{n+2}, 19861y_{2n+3} - 1241x_{2n+4} + 264)$
14	$4Y - 17X^2 = 64$	$(19861x_{n+3} - 4817y_{n+3}, 19861y_{2n+4} - 81889x_{2n+4} + 8)$
15	$544Y - 17X^2 = 1183744$	$(301y_{n+3} - 19861y_{n+2}, 81889y_{2n+3} - 1241y_{2n+4} + 1088)$

G. Some Special Cases Of The Solutions Are Given Below

- $P_y^{10}(t_{3,x+1})^2 = 153P_x^6(t_{3,y})^2 + 8(t_{3,y})^2(t_{3,x+1})^2$
- $9P_y^6(t_{3,x})^2 = 17P_x^{10}(t_{3,y+1})^2 + 8(t_{3,x})^2(t_{3,y+1})^2$
- $P_y^{10}(t_{3,2x-2})^2 = 17(6P_{x-1}^4)^2(t_{3,y})^2 + 8(t_{3,y})^2(t_{3,2x-2})^2$
- $36P_{y-1}^8(t_{3,x})^2 = 17P_x^{10}(t_{3,2y-2})^2 + 8(t_{3,x})^2(t_{3,2y-2})^2$
- $9P_y^6(t_{3,2x-2})^2 = 17(36P_{x-1}^8)(t_{3,y+1})^2 + 8(t_{3,2x-2})^2(t_{3,y+1})^2$
- $(6P_{y-1}^4)^2(t_{3,x+1})^2 = 17(3P_x^3)^2(t_{3,2y-2})^2 + 8(t_{3,x+1})^2(t_{3,2y-2})^2$

III.CONCLUSIONS

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2 = 17x^2 + 8$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.



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