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On the Positive Pell Equation $y^2 = 17x^2 + 8$

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Abstract: The binary quadratic Diophantine equation $y^2 = 17x^2 + 8$ is analyzed for its non-zero distinct integral solutions. A few interesting relations among the solutions are given. Further, employing the solutions have obtained solutions of other choices of hyperbolas and parabolas.
Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4] .For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 17x^2 + 8$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented

II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 17x^2 + 8 \qquad (1)$$

The smallest positive integer solutions of (1) are,

$$x_0 = 1, y_0 = 5 \quad D = 17$$

consider the pellian equation is

$$y^2 = 17x^2 + 1 \qquad (2)$$

The initial solution of pellian equation is

$$\tilde{x}_0 = 8, \tilde{y}_0 = 33,$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{17}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (33 + 8\sqrt{17})^{n+1} + (33 - 8\sqrt{17})^{n+1}$$

$$g_n = (33 + 8\sqrt{17})^{n+1} - (33 - 8\sqrt{17})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{17}} g_n$$

$$y_{n+1} = \frac{5}{2} f_n + \frac{17}{2\sqrt{17}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$\mathbf{x_{n+3} - 66x_{n+2} + x_{n+1} = 0}$$

$$\mathbf{y_{n+3} - 66y_{n+2} + y_{n+1} = 0}$$

Where n=0,1,2,3..........

Some numerical examples of x_n and y_n satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	1	5
1	73	301
2	4817	19861
3	317849	1310525
4	20973217	86474789

From the above table, we observe some interesting relations among the solutions which are presented below.

A. Both x_n and y_n values are odd .

B. Each Of The Following Expression Is A Nasty Number

1) $\frac{3}{2}[8+5y_{2n+2}-17x_{2n+2}]$

2) $\frac{3}{16}[64+5x_{2n+3}-301x_{2n+2}]$

3) $\frac{1}{352}[4224+5x_{2n+4}-19861x_{2n+2}]$

4) $\frac{1}{22}[264+5y_{2n+3}-1241x_{2n+2}]$

5) $\frac{3}{4354}[17416+5y_{2n+4}-81889x_{2n+2}]$

6) $\frac{1}{22}[264+301y_{2n+2}-17x_{2n+3}]$

7) $\frac{3}{4354}[17416+19861y_{2n+2}-17x_{2n+4}]$

8) $\frac{3}{272}[1088+1241y_{2n+2}-17y_{2n+3}]$

9) $\frac{1}{5984}[71808+81889y_{2n+2}-17y_{2n+4}]$

10) $\frac{3}{2}[8+301y_{2n+3}-1241x_{2n+3}]$

11) $\frac{3}{16}[64+301x_{2n+4}-19861x_{2n+3}]$

12) $\frac{1}{22}[264+301y_{2n+4}-81889x_{2n+3}]$

13) $\frac{1}{22}[264+19861y_{2n+3}-1241x_{2n+4}]$

14) $\frac{3}{2}[8+19861y_{2n+4}-81889x_{2n+4}]$

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15) $\frac{3}{272}\left[1088+81889y_{2n+3}-1241y_{2n+4}\right]$

C. Each Of The Following Expressions Is A Cubical Integer

1) $\frac{1}{4}\left[5y_{3n+3}-17x_{3n+3}+15y_{n+1}-51x_{n+1}\right]$

2) $\frac{1}{32}\left[5x_{3n+4}-301x_{3n+3}+15x_{n+2}-903x_{n+1}\right]$

3) $\frac{1}{2112}\left[5x_{3n+5}-19861x_{3n+3}+15x_{n+3}-59583x_{n+1}\right]$

4) $\frac{1}{132}\left[5y_{3n+4}-1241x_{3n+3}+15y_{n+2}-3723x_{n+1}\right]$

5) $\frac{1}{8708}\left[5y_{3n+5}-81889x_{3n+3}+15y_{n+3}-245667x_{n+1}\right]$

6) $\frac{1}{132}\left[301y_{3n+3}-17x_{3n+4}+903y_{n+1}-51x_{n+2}\right]$

7) $\frac{1}{8708}\left[19861y_{3n+3}-17x_{3n+5}+59583y_{n+1}-51x_{n+3}\right]$

8) $\frac{1}{544}\left[1241y_{3n+3}-17y_{3n+4}+3723y_{n+1}-51y_{n+2}\right]$

9) $\frac{1}{35904}\left[81889y_{3n+3}-17y_{3n+5}+245667y_{n+1}-51y_{n+3}\right]$

10) $\frac{1}{32}\left[301x_{3n+5}-19861x_{3n+4}+903x_{n+3}-59583x_{n+2}\right]$

11) $\frac{1}{4}\left[301y_{3n+4}-1241x_{3n+4}+903y_{n+2}-3723x_{n+2}\right]$

12) $\frac{1}{132}\left[301y_{3n+5}-81889x_{3n+4}+903y_{n+3}-245667x_{n+2}\right]$

13) $\frac{1}{132}\left[19861y_{3n+4\backslash}-1241x_{3n+5}+59583y_{n+2}-3723x_{n+3}\right]$

14) $\frac{1}{4}\left[19861y_{3n+5\backslash}-81889x_{3n+5}+59583y_{n+3}-245667x_{n+3}\right]$

15) $\frac{1}{544}\left[81889y_{3n+4\backslash}-1241y_{3n+5}+245667y_{n+2}-3723y_{n+3}\right]$

D. Each Of The Following Expressions Is A Biquadratic Integer

1) $\frac{1}{4}\left[5y_{4n+4}-17x_{4n+4}+20y_{2n+2}-68x_{2n+2}+24\right]$

2) $\frac{1}{32}\left[5x_{4n+5}-301x_{4n+4}+20x_{2n+3}-1204x_{2n+2}+192\right]$

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3) $\frac{1}{2112}\left[5x_{4n+6} - 19861x_{4n+4} + 20x_{2n+4} - 79444x_{2n+2} + 12672\right]$

4) $\frac{1}{132}\left[5y_{4n+5} - 1241x_{4n+4} + 20y_{2n+3} - 4964x_{2n+2} + 792\right]$

5) $\frac{1}{8708}\left[5y_{4n+6} - 81889x_{4n+4} + 20y_{2n+4} - 327556x_{2n+2} + 52248\right]$

6) $\frac{1}{132}\left[301y_{4n+4} - 17x_{4n+5} + 1204y_{2n+2} - 68x_{2n+3} + 792\right]$

7) $\frac{1}{8708}\left[19861y_{4n+4} - 17x_{4n+6} + 79444y_{2n+2} - 68x_{2n+4} + 52248\right]$

8) $\frac{1}{544}\left[1241y_{4n+4} - 17y_{4n+5} + 4964y_{2n+2} - 68y_{2n+3} + 3264\right]$

9) $\frac{1}{35904}\left[81889y_{4n+4} - 17y_{4n+6} + 327556y_{2n+2} - 68y_{2n+4} + 215424\right]$

10) $\frac{1}{32}\left[301x_{4n+6} - 19861x_{4n+5} + 1204x_{2n+4} - 79444x_{2n+3} + 192\right]$

11) $\frac{1}{4}\left[301y_{4n+5} - 1241x_{4n+5} + 1204y_{2n+3} - 4964x_{2n+3} + 24\right]$

12) $\frac{1}{132}\left[301y_{4n+6} - 81889x_{4n+5} + 1204y_{2n+4} - 327556x_{2n+3} + 792\right]$

13) $\frac{1}{132}\left[19861y_{4n+5} - 1241x_{4n+6} + 79444y_{2n+3} - 4964x_{2n+4} + 792\right]$

14) $\frac{1}{4}\left[19861y_{4n+6} - 81889x_{4n+6} + 79444y_{2n+4} - 327556x_{2n+4} + 24\right]$

15) $\frac{1}{544}\left[81889y_{4n+5} - 1241y_{4n+6} + 327556y_{2n+3} - 4964y_{2n+4} + 3264\right]$

E. Each Of The Following Expression Is A Quintic Integer

1) $\frac{1}{4}\left[5y_{5n+5} - 17x_{5n+5} + 25y_{3n+3} - 85x_{3n+3} + 50y_{n+1} - 170x_{n+1}\right]$

2) $\frac{1}{32}\left[5x_{5n+6} - 301x_{5n+5} + 25x_{3n+4} - 1505x_{3n+3} + 50x_{n+2} - 3010x_{n+1}\right]$

3) $\frac{1}{2112}\left[5x_{5n+7} - 19861x_{5n+5} + 25x_{3n+5} - 99305x_{3n+3} + 50x_{n+3} - 198610x_{n+1}\right]$

4) $\frac{1}{132}\left[5y_{5n+6} - 1241x_{5n+5} + 25y_{3n+4} - 6205x_{3n+3} + 50y_{n+2} - 12410x_{n+1}\right]$

5) $\frac{1}{8708}\left[5y_{5n+7} - 81889x_{5n+5} + 25y_{3n+5} - 409445x_{3n+3} + 50y_{n+3} - 818890x_{n+1}\right]$

6) $\frac{1}{132}\left[301y_{5n+5} - 17x_{5n+6} + 1505y_{3n+3} - 85x_{3n+4} + 3010y_{n+1} - 170x_{n+2}\right]$

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7) $\frac{1}{544}[1241y_{5n+5} - 17y_{5n+6} + 6205y_{3n+3} - 85y_{3n+4} + 12410y_{n+1} - 170y_{n+2}]$

8) $\frac{1}{2112}[4817y_{5n+5} - y_{5n+7} + 24085y_{3n+3} - 5y_{3n+5} + 48170y_{n+1} - 10y_{n+3}]$

9) $\frac{1}{32}[301x_{5n+7} - 19861x_{5n+6} + 1505x_{3n+5} - 99305x_{3n+4} + 3010x_{n+3} - 198610x_{n+2}]$

10) $\frac{1}{4}[301y_{5n+6} - 1241x_{5n+6} + 1505y_{3n+4} - 6205x_{3n+4} + 3010y_{n+2} - 12410x_{n+2}]$

11) $\frac{1}{132}[301y_{5n+7} - 81889x_{5n+6} + 1505y_{3n+5} - 409445x_{3n+4} + 3010y_{n+3} - 818890x_{n+2}]$

12) $\frac{1}{132}[19861y_{5n+6} - 1241x_{5n+7} + 99305y_{3n+4} - 6205x_{3n+5} + 198610y_{n+2} - 12410x_{n+3}]$

13) $\frac{1}{4}[19861y_{5n+7} - 81889x_{5n+7} + 99305y_{3n+5} - 409445x_{3n+5} + 198610y_{n+3} - 818890x_{n+3}]$

14) $\frac{1}{544}[81889y_{5n+6} - 1241y_{5n+7} + 409445y_{3n+4} - 6205y_{3n+5} + 818890y_{n+2} - 12410y_{n+3}]$

F. *Relations Among The Solutions Are Given Below*

1) $x_{n+2} = 8y_{n+1} + 33x_{n+1}$
2) $x_{n+3} = 528y_{n+1} + 2177x_{n+1}$
3) $y_{n+2} = 33y_{n+1} + 136x_{n+1}$
4) $y_{n+3} = 2177y_{n+1} + 8976x_{n+1}$
5) $x_{n+3} = 66x_{n+2} - x_{n+1}$
6) $8y_{n+2} = 33x_{n+2} - x_{n+1}$
7) $8y_{n+3} = 2177x_{n+2} - 33x_{n+1}$
8) $16y_{n+2} = x_{n+3} - x_{n+1}$
9) $528y_{n+3} = 2177x_{n+3} - x_{n+1}$
10) $33y_{n+3} = 2177y_{n+2} + 136x_{n+1}$
11) $33x_{n+3} = 8y_{n+1} + 2177x_{n+2}$
12) $33y_{n+2} = y_{n+1} + 136x_{n+2}$
13) $y_{n+3} = y_{n+1} + 272x_{n+2}$
14) $2177y_{n+2} = 33y_{n+1} + 136x_{n+3}$
15) $2177y_{n+3} = y_{n+1} + 8976x_{n+3}$
16) $y_{n+3} = 66y_{n+2} - y_{n+1}$
17) $8y_{n+2} = x_{n+3} - 33x_{n+2}$
18) $8y_{n+3} = 33x_{n+3} - x_{n+2}$

19) $y_{n+3} = 33y_{n+2} + 136x_{n+2}$

20) $33y_{n+3} = y_{n+2} + 136x_{n+3}$

III. REMARKABLE OBSERVATIONS

A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below:

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 17X^2 = 64$	$(5x_{n+1} - y_{n+1}, 5y_{n+1} - 17x_{n+1})$
2	$Y^2 - 17X^2 = 4096$	$(73x_{n+1} - x_{n+2}, 5x_{n+2} - 301x_{n+1})$
3	$Y^2 - 17X^2 = 17842176$	$(4817x_{n+1} - x_{n+3}, 5x_{n+3} - 19861x_{n+1})$
4	$Y^2 - 17X^2 = 69696$	$(301x_{n+1} - y_{n+2}, 5y_{n+2} - 1241x_{n+1})$
5	$Y^2 - 17X^2 = 303317056$	$(19861x_{n+1} - y_{n+3}, 5y_{n+3} - 81889x_{n+1})$
6	$Y^2 - 17X^2 = 69696$	$(5x_{n+2} - 73y_{n+1}, 301y_{n+1} - 17x_{n+2})$
7	$Y^2 - 17X^2 = 303317056$	$(5x_{n+3} - 4817y_{n+1}, 19861y_{n+1} - 17x_{n+3})$
8	$Y^2 - 17X^2 = 1183744$	$(5y_{n+2} - 301y_{n+1}, 1241y_{n+1} - 17y_{n+2})$
9	$Y^2 - 17X^2 = 5156388864$	$(5y_{n+3} - 19861y_{n+1}, 81889y_{n+1} - 17y_{n+3})$
10	$Y^2 - 17X^2 = 4096$	$(4817x_{n+2} - 73x_{n+3}, 301x_{n+3} - 19861x_{n+2})$
11	$Y^2 - 17X^2 = 64$	$(301x_{n+2} - 73y_{n+2}, 301y_{n+2} - 1241x_{n+2})$
12	$Y^2 - 17X^2 = 69696$	$(19861x_{n+2} - 73y_{n+3}, 301y_{n+3} - 81889x_{n+2})$
13	$Y^2 - 17X^2 = 69696$	$(301x_{n+3} - 4817y_{n+2}, 19861y_{n+2} - 1241x_{n+3})$
14	$Y^2 - 17X^2 = 64$	$(19861x_{n+3} - 4817y_{n+3}, 19861y_{n+3} - 81889x_{n+3})$
15	$Y^2 - 17X^2 = 1183744$	$(301y_{n+3} - 19861y_{n+2}, 81889y_{n+2} - 1241y_{n+3})$

B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below:

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$4Y-17X^2=64$	$(5x_{n+1}-y_{n+1}, 5y_{2n+2}-17x_{2n+2}+8)$
2	$32Y-17X^2=4096$	$(73x_{n+1}-x_{n+2}, 5x_{2n+3}-301x_{2n+2}+64)$
3	$2112Y-17X^2=17842176$	$(4817x_{n+1}-x_{n+3}, 5x_{2n+4}-19861x_{2n+2}+4224)$
4	$132Y-17X^2=69696$	$(301x_{n+1}-y_{n+2}, 5y_{2n+3}-1241x_{2n+2}+264)$
5	$8708Y-17X^2=303317056$	$(19861x_{n+1}-y_{n+3}, 5y_{2n+4}-81889x_{2n+2}+17416)$
6	$132Y-17X^2=69696$	$(5x_{n+2}-73y_{n+1}, 301y_{2n+2}-17x_{2n+3}+264)$
7	$8708Y-17X^2=303317056$	$(5x_{n+3}-4817y_{n+1}, 19861y_{2n+2}-17x_{2n+4}+17416)$
8	$544Y-17X^2=1183744$	$(5y_{n+2}-301y_{n+1}, 1241y_{2n+2}-17y_{2n+3}+1088)$
9	$35904Y-17X^2=5156388864$	$(5y_{n+3}-19861y_{n+1}, 81889y_{2n+2}-17y_{2n+4}+71808)$
10	$32Y-17X^2=4096$	$(4817x_{n+2}-73x_{n+3}, 301x_{2n+4}-19861x_{2n+3}+64)$
11	$4Y-17X^2=64$	$(301x_{n+2}-73y_{n+2}, 301y_{2n+3}-1241x_{2n+3}+8)$
12	$132Y-17X^2=69696$	$(19861x_{n+2}-73y_{n+3}, 301y_{2n+4}-81889x_{2n+3}+264)$
13	$132Y-17X^2=69696$	$(301x_{n+3}-4817y_{n+2}, 19861y_{2n+3}-1241x_{2n+4}+264)$
14	$4Y-17X^2=64$	$(19861x_{n+3}-4817y_{n+3}, 19861y_{2n+4}-81889x_{2n+4}+8)$
15	$544Y-17X^2=1183744$	$(301y_{n+3}-19861y_{n+2}, 81889y_{2n+3}-1241y_{2n+4}+1088)$

G. Some Special Cases Of The Solutions Are Given Below

1) $P_y^{10}(t_{3,x+1})^2=153P_x^6(t_{3,y})^2+8(t_{3,y})^2(t_{3,x+1})^2$

2) $9P_y^6(t_{3,x})^2=17P_x^{10}(t_{3,y+1})^2+8(t_{3,x})^2(t_{3,y+1})^2$

3) $P_y^{10}(t_{3,2x-2})^2=17(6P_{x-1}^4)^2(t_{3,y})^2+8(t_{3,y})^2(t_{3,2x-2})^2$

4) $36P_{y-1}^8(t_{3,x})^2=17P_x^{10}(t_{3,2y-2})^2+8(t_{3,x})^2(t_{3,2y-2})^2$

5) $9P_y^6(t_{3,2x-2})^2=17(36P_{x-1}^8)(t_{3,y+1})^2+8(t_{3,2x-2})^2(t_{3,y+1})^2$

6) $(6P_{y-1}^4)^2(t_{3,x+1})^2=17(3P_x^3)^2(t_{3,2y-2})^2+8(t_{3,x+1})^2(t_{3,2y-2})^2$

III.CONCLUSIONS

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2=17x^2+8$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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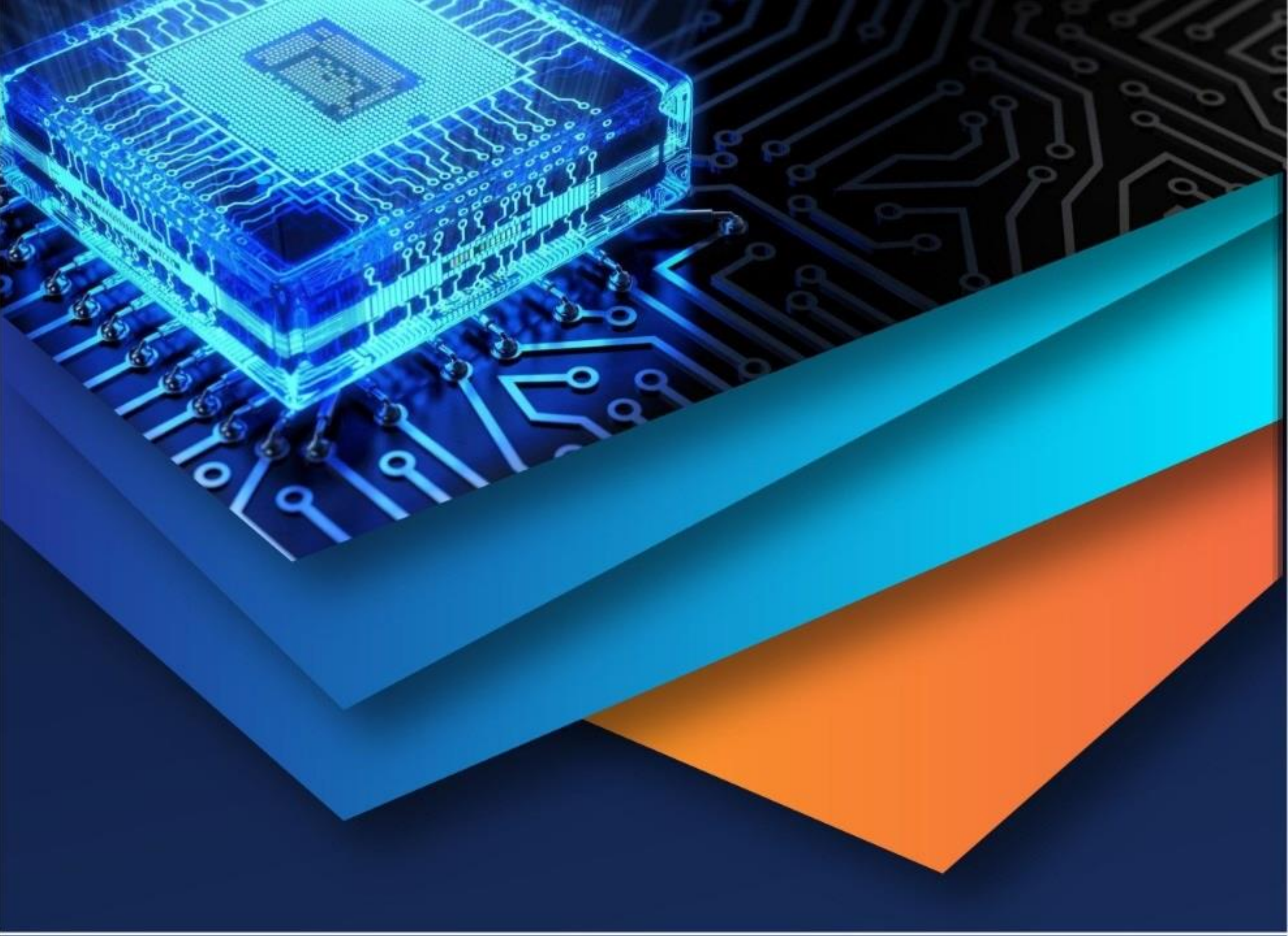

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