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# Equivalence of Two Summability Methods for Improper Integral

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**Abstract:** In this paper, we introduce the concept of  $|N, p; \delta, \mu|_k, k \geq 1$  summability of improper integrals and by this definition we prove a theorem that generalizes theorems of Orhan, Ozgen and Acharya.

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## I. INTRODUCTION

Let  $f$  be a real valued function which is continuous on  $[0, \infty)$ . Let  $s(x) = \int_0^x f(t) dt$ . We denote the Cesaro mean of  $s(x)$  by

$\sigma(x)$ . The improper integral  $\int_0^\infty f(t) dt$  is said to be integrable  $|C, 1|_k, k \geq 1$ , in the sense of Flett[3] if

$$(1.1) \quad \int_0^\infty x^{k-1} |\sigma'(x)|^k dx = \int_0^\infty \frac{|v(x)|^k}{x} dx$$

is convergent, where  $v(x) = \frac{1}{x} \int_0^\infty t f(t) dt$  usually called a generator of the integral  $\int_0^\infty f(t) dt$ .

Let  $p(x)$  be a non-decreasing real valued function on  $[0, \infty)$ . We define

$$P(x) = \int_0^x p(t) dt, p(x) \neq 0, p(0) = 0.$$

Then we define the Norlund mean of  $s(x)$  by

$$\sigma_p(x) = \frac{1}{P(x)} \int_0^x p(t) s(t) dt.$$

We say that the improper integral  $\int_0^\infty f(t) dt$  is summable  $|N, p|_k, k \geq 1$ , if

$$(1.2) \quad \int_0^\infty x^{k-1} |\sigma'_p(x)|^k dx$$

is convergent. In the special case if we take  $p(x) = 1$  for all values of  $x$ , then  $|N, p|_k$

summability reduces to  $|C, 1|_k$  summability of improper integrals.

For any two functions  $f$  and  $g$ , it is customary to write  $g(x) = O(f(x))$ , if there exist  $K$  and  $N$ , for every

$x > N$ ,  $\left| \frac{g(x)}{f(x)} \right| \leq K$ . Clearly the difference between  $s(x)$  and its  $n$ th weighted mean  $\sigma_p(x)$ , which is called the weighted

Kronecker identity, is given by the identity

$$(1.3) \quad s(x) - \sigma_p(x) = v_p(x),$$

where

$$v_p(x) = \frac{1}{P(x)} \int_0^\infty p(u) f(u) du.$$

We note that if we take  $p(x) = 1$ , for all values of  $x$ , then we have the following identity ( see [2])

$$s(x) - \sigma_p(x) = v(x).$$

Since

$$\sigma'_p(x) = \frac{p(x)}{P(x)} v_p(x),$$

condition (1.3) can be written as

$$(1.4) \quad s(x) = v_p(x) + \int_0^x \frac{p(u)}{P(u)} v_p(u) du.$$

In view of the identity (1.4), the function  $v_p(x)$  is called the generator function of  $s(x)$ .

Condition (1.1) can also be written as

$$(1.5) \quad \int_0^\infty x^{k-1} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx$$

is convergent. It may be noted that for infinite series, an analogous definition was introduced by Orhan [4]. The improper integral

$\int_0^\infty f(t) dt$  is summable  $|N, p; \delta|_k, k \geq 1, 0 \leq \delta k < 1$ , if

$$(1.6) \quad \int_0^\infty x^{\delta k + k - 1} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx$$

and summable  $|N, p; \delta, \mu|_k, k \geq 1, 0 \leq \delta k < 1, \mu \geq 1$ , if

$$(1.7) \quad \int_0^\infty x^{\mu(\delta k + k - 1)} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx.$$

## II. KNOWN RESULTS

Dealing with  $|R, p_n|_k$  and  $|R, q_n|_k$  summability methods, Orhan [4] proved the following theorem:

1) *Theorem 2.1.* The  $|R, p_n|_k, (k \geq 1)$  summability implies the  $|R, q_n|_k, (k \geq 1)$  summability provided that

a)  $nq_n = O(Q_n)$ ,

b)  $P_n = O(np_n)$  and

c)  $Q_n = O(nq_n)$ .

Subsequently, dealing with summability of improper integrals, Ozgen has established the following theorem:

2) *Theorem 2.2.* [5] Let  $p(x)$  and  $q(x)$  be non-decreasing real valued functions on  $[0, \infty)$  such that as  $x \rightarrow \infty$

a)  $xq(x) = O(Q(x))$ ,

b)  $P(x) = O(xp(x))$ ,

and

c)  $Q(x) = O(xq(x))$

If  $\int_0^\infty f(t)dt$  is summable  $|R, p|_k$ , then it is also summable  $|R, q|_k, (k \geq 1)$ .

Extending the result of Ozgen, Acharya[1] has established the following result.

3) *Theorem 2.3.* Let  $p(x)$  and  $q(x)$  be non-decreasing real valued functions on  $[0, \infty)$  such that as  $x \rightarrow \infty$

a)  $xq(x) = O(Q(x))$ ,

b)  $P(x) = O(xp(x))$ ,

c)  $Q(x) = O(xq(x))$ ,

d)  $\int_t^m \frac{x^{\delta k} q(x)}{Q^2(x)} dx = O\left(\frac{t^{\delta k}}{Q(t)}\right)$

and

e)  $\int_0^m t^{\delta k - 1} |v_p(t)|^k dt = O(1)$ .

If  $\int_0^\infty f(t)dt$  is summable  $|N, p; \delta|_k$ , then it is also summable  $|N, q, \delta|_k, (k \geq 1)$ .

However, extending the above result, in the present paper we establish the following theorem :

### III. MAIN RESULT

1) *Theorem 3.1.* Let  $p(x)$  and  $q(x)$  be non-decreasing real valued functions on  $[0, \infty)$  such that as  $x \rightarrow \infty$

a)  $xq(x) = O(Q(x))$ ,

b)  $P(x) = O(xp(x))$ ,

c)  $Q(x) = O(xq(x))$ ,

d)  $\int_t^m \frac{x^{\mu(\delta k + k - 1) - k + 1} q(x)}{Q^2(x)} dx = O\left(\frac{t^{\mu(\delta k + k - 1) - k + 1}}{Q(t)}\right)$

and

e)  $\int_0^m t^{\mu(\delta k + k - 1) - k} |v_p(t)|^k dt = O(1)$ .

If  $\int_0^\infty f(t)dt$  is summable  $|N, p; \delta, \mu|_k$ , then it is also summable  $|N, q, \delta, \mu|_k, (k \geq 1)$ .

### IV. PROOF OF THE THEOREM

Let  $\sigma_p(x)$  and  $\sigma_q(x)$  be the functions of  $(N, p)$  and  $(N, q)$  means of the integral  $\int_0^\infty f(t)dt$ . Since  $\int_0^\infty f(t)dt$  is summable  $|N, p; \delta, \mu|_k, k \geq 1, 0 \leq \delta k < 1, \mu \geq 1$ , we can write

$$\int_0^\infty x^{\mu(\delta k+k-1)} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx$$

is convergent. Differentiating the equation (1.4), we have

$$f(x) = v_p'(x) + \frac{p(x)}{P(x)} v_p(x) .$$

By definition, we obtain

$$\sigma_q(x) = \frac{1}{Q(x)} \int_0^x q(t) f(t) dt = \frac{1}{Q(x)} \int_0^x (Q(x) - Q(t)) f(t) dt$$

and

$$\begin{aligned} \sigma_q'(x) &= \frac{q(x)}{Q^2(x)} \int_0^x Q(t) f(t) dt \\ &= \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \left[ v_p'(t) + \frac{p(t)}{P(t)} v_p(t) \right] dt \\ &= \frac{q(x)}{Q^2(x)} \int_0^x Q(t) v_p'(t) dt + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t) dt \end{aligned}$$

Integrating by parts of the first statement, we have

$$\begin{aligned} \sigma_q'(x) &= \frac{q(x)}{Q^2(x)} \left[ Q(x) v_p(x) - \int_0^x q(t) v_p(t) dt \right] + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t) dt \\ &= \frac{q(x)}{Q^2(x)} v_p(x) + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t) dt - \frac{q(x)}{Q^2(x)} \int_0^x q(t) v_p(t) dt \\ &= \sigma_{q,1}(x) + \sigma_{q,2}(x) + \sigma_{q,3}(x), \text{ say.} \end{aligned}$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\int_0^m x^{\mu(\delta k+k-1)} |\sigma_{q,r}(x)|^k dx = O(1) \text{ as } m \rightarrow \infty, \text{ for } r = 1, 2, 3 .$$

Using conditions (7.3.1.1) and (7.3.1.2), we have

$$\begin{aligned} \int_0^m x^{\mu(\delta k+k-1)} |\sigma_{q,1}(x)|^k dx &= \int_0^m x^{\mu(\delta k+k-1)} \left| \frac{q(x)}{Q(x)} v_p(x) \right|^k dx \\ &= \int_0^m x^{\mu(\delta k+k-1)} \left( \frac{q(x)}{Q(x)} \right)^k |v_p(x)|^k dx \\ &= O(1) \int_0^m x^{\mu(\delta k+k-1)} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx \\ &= O(1) \int_0^m x^{\mu(\delta k+k-1)} |\sigma_p'(x)|^k dx \\ &= O(1) , \text{ as } m \rightarrow \infty , \end{aligned}$$

by virtue of the hypotheses of theorem 3.1.

Applying Holder's inequality with  $k > 1$ , we get

$$\int_0^m x^{\mu(\delta k+k-1)} |\sigma_{q,2}(x)|^k dx = O(1) \int_0^m x^{\mu(\delta k+k-1)} \left( \frac{q(x)}{Q^2(x)} \right)^k \left( \int_0^x \frac{Q(t) p(t)}{P(t)} |v_p(t)| dt \right)^k dx$$

$$\begin{aligned}
 &= O(1) \int_0^m \frac{x^{\mu(\delta k+k-1)-k+1} q(x)}{Q^{k+1}(x)} \left( \int_0^x \frac{Q(t)p(t)}{P(t)} |v_p(t)| dt \right)^k dx \\
 &= O(1) \int_0^m \frac{x^{\mu(\delta k+k-1)-k+1} q(x)}{Q^2(x)} \left( \int_0^x \left( \frac{Q(t)}{q(t)} \right)^k q(t) \left( \frac{P(t)}{P(t)} \right)^k |v_p(t)|^k dt \right) x \left( \frac{1}{Q(x)} \int_0^x q(t) dt \right)^{k-1} dx \\
 &= O(1) \int_0^m t^k q(t) \left( \frac{P(t)}{P(t)} \right)^k |v_p(t)|^k dt \int_t^m \frac{x^{\mu(\delta k+k-1)-k+1} q(x)}{Q^2(x)} dx \\
 &= O(1) \int_0^m t^{\mu(\delta k+k-1)+1} \frac{q(t)}{Q(t)} \left( \frac{P(t)}{P(t)} \right)^k |v_p(t)|^k dt \\
 &= O(1) \int_0^m t^{\mu(\delta k+k-1)} \left( \frac{P(t)}{P(t)} \right)^k |v_p(t)|^k dt \\
 &= O(1) \text{ as } m \rightarrow \infty,
 \end{aligned}$$

by virtue of the hypotheses of theorem 7.3.1.

Finally, again by Holder's inequality with  $k > 1$ , we have

$$\begin{aligned}
 \int_0^m x^{\mu(\delta k+k-1)} |\sigma_{q,3}(x)|^k dx &= O(1) \int_0^m x^{\mu(\delta k+k-1)} \left( \frac{q(x)}{Q^2(x)} \right)^k \left( \int_0^x q(t) |v_p(t)|^k dt \right)^k dx \\
 &= O(1) \int_0^m \frac{x^{\mu(\delta k+k-1)-k+1} q(x)}{Q^2(x)} \left( \int_0^x q(t) |v_p(t)|^k dt \right) x \left( \frac{1}{Q(x)} \int_0^x q(t) dt \right)^{k-1} dx \\
 &= O(1) \int_0^m q(t) |v_p(t)|^k dt \int_t^m \frac{x^{\mu(\delta k+k-1)-k+1} q(x)}{Q^2(x)} dx \\
 &= O(1) \int_0^m \frac{t^{\mu(\delta k+k-1)-k+1} q(t)}{Q(t)} |v_p(t)|^k dt \\
 &= O(1) \int_0^m t^{\mu(\delta k+k-1)-k} |v_p(t)|^k dt \\
 &= O(1) \text{ as } m \rightarrow \infty,
 \end{aligned}$$

by virtue of the hypotheses of theorem 3.1.

This completes the proof of the theorem.

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