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On Q^* - Homeomorphisms in Topological Spaces

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Abstract: In this paper, we first introduce a new class of closed map called Q^* - closed map. Moreover, we introduce a new class of homeomorphism called Q^* - homeomorphism, which are weaker than homeomorphism. We also introduce Q^* - homeomorphisms and prove that the set of all Q^* -homeomorphisms forms a group under the operation of composition of maps. We prove that Q^* - homeomorphism and Q^* - homeomorphism are independent.

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I. INTRODUCTION

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijective map $f: X \rightarrow Y$ when both f and f^{-1} are continuous. It is well known that as Janich [[8], p.13] says correctly: homeomorphisms play the same role in topology that linear isomorphism's play in linear algebra, or that biholomorphic maps play in function theory, or group isomorphism's in group theory, or isometrics in Riemannian geometry. In 1995, Maki, Devi and Balachandran [4] introduced the concepts of semi - generalized homeomorphisms and generalized semi-homeomorphisms and studied some semi topological properties. Devi and Balachandran [3] introduced a generalization of α -homeomorphism in 2001. Recently, Devi, Vigneshwaran, Vadivel and Vairamanickam [5,23] introduced $g^*\alpha$ -homeomorphisms and $rg\alpha$ - homeomorphisms and obtained some topological properties. In this paper, I first introduce a new class of closed map called Q^* - closed map. Moreover, we introduce a new class of homeomorphism called Q^* - homeomorphism, which are weaker than homeomorphism. We also introduce Q^* - homeomorphisms and prove that the set of all Q^* -homeomorphisms forms a group under the operation of composition of maps. We prove that Q^* - homeomorphism and Q^* - homeomorphism are independent.

II. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) , $cl(A)$, $int(A)$ and A^c denote the closure of A , the interior of A and the complement of A in X , respectively. We recall the following definitions and some results, which are used in the sequel.

Definition 2.1 A set A is said to be semi open if there exists an open set G such that $G \subset A \subset cl(G)$ or $A \subset cl(int(A))$. The set A is semi closed if $int(cl(A)) \subset A$.

Definition 2.2 The function $f: X \rightarrow Y$ is said to be irresolute if $f^{-1}(A)$ is a semi open set whenever A is a semi open set in Y .

Definition 2.3 A subset A of a space (X, τ) is called pre-closed if $cl(int(A)) \subset A$.

Definition 2.4 A space X is said to be Q^* - regular if for every Q^* - closed set F and a point $x \notin F$, there exist disjoint Q^* - open sets U and V such that $x \in U$ and $F \subseteq V$.

Definition 2.5[15] A subset A of a topological space (X, τ) is called a Q^* - closed if $int(A) = \emptyset$ and A is closed

III. Q^* HOMEOMORPHISMS

Q^* - closed sets in topological spaces were introduced by M. Murugalingam and N. Lalitha in 2010. In this section we study the properties of Q^* - homeomorphisms, Q^* compact spaces are discussed.

Definition 3.1 A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is pre Q^* - closed if $f(U)$ is Q^* closed in Y for every Q^* closed set in X .

Definition 3.2 A bijection $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* - homeomorphism if f is Q^* irresolute and pre Q^* - open.

Example 3.1 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$ and $\sigma = \{\emptyset, Y, \{b, c\}\}$. Let $f: X \rightarrow Y$ be the identity map. Hence f is Q^* - homeomorphism.

Proposition 3.1 Every Q^* - closed set is α -closed and hence pre-closed.

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Definition 3.3 A space X is said to be Q^* -compact if every cover of X by Q^* -open sets has a finite sub cover.

Definition 3.4 A map $f: X \rightarrow Y$ is called Q^* -closed map if the image of each Q^* -closed in (X, τ) is closed in (Y, σ) .

Example 3.2 (1) Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}\}$ and $\sigma = \{\emptyset, Y, \{b\}, \{b, c\}\}$. Then $\emptyset, \{a, c\}$ are Q^* -closed in X . Let $f: X \rightarrow Y$ be the identity map. Then $f(\emptyset) = \emptyset$, $f(\{a, c\}) = \{a, c\}$. Since $\emptyset, \{a, c\}$ are closed in Y . Therefore, f is Q^* -closed map.

Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is not a Q^* -closed map.

The next theorem shows that normality is preserved under continuous

Definition 3.5 Let x be a point of (X, τ) and V be a subset of X . Then V is called a Q^* -neighborhood of x in (X, τ) if there exists a Q^* -open set U of (X, τ) such that $x \in U \subset V$.

Theorem 3.1 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -continuous, Q^* -closed map from a Q^* -normal space (X, τ) onto a space (Y, σ) then (Y, σ) is Q^* -normal.

Proof. Since every Q^* -closed map is Q^* -closed.

Analogous to a Q^* -closed map, we define a Q^* -open map as follows:

Definition 3.6 A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be a Q^* -open map if the image $f(A)$ is Q^* -open in (Y, σ) for each open set A in (X, τ) .

Proposition 3.2 For any bijection $f: (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:

- (i) $f^{-1}: (Y, \sigma) \rightarrow (X, \tau)$ is Q^* -continuous,
- (ii) f is a Q^* -open map and
- (iii) f is a Q^* -closed map.

Proof.

Step 1: (a) \Rightarrow (b)

Let V be a Q^* -open set in X . Then $X - V$ is Q^* -closed in X . Since f^{-1} is Q^* -continuous, $(f^{-1})^{-1}(X - V) = f(X - V) = Y - f(V)$ is closed in Y . Then $f(V)$ is open in Y . Hence f is a Q^* -open map.

Step 2: (b) \Rightarrow (c)

Let f be a Q^* -open map. Let U be a Q^* -closed set in X . Then $X - U$ is Q^* -open in X . Since f is Q^* -open, $f(X - U) = Y - f(U)$ is open in Y . Then $f(U)$ is closed in Y . Hence f is a Q^* -closed.

Step 3: (c) \Rightarrow (a)

Let V be Q^* -closed set in X . Since $f: X \rightarrow Y$ is Q^* -closed, $f(V)$ is closed in Y . That is $(f^{-1})^{-1}(V)$ is closed in Y . Hence f^{-1} is Q^* -continuous. ■

Proposition 3.3 Let $f: X \rightarrow Y$ be a bijective and Q^* -continuous map. Then the following statements are equivalent.

- (a) f is a Q^* -open map.
- (b) f is a Q^* -homeomorphism.
- (c) f is a Q^* -closed map.

Proof.

Step 1: (a) \Rightarrow (b)

Given f is bijective, Q^* -continuous map and Q^* -open map. Hence f is Q^* -homeomorphism.

Step 2: (b) \Rightarrow (c)

Suppose (b) holds. Let F be a closed set in (X, τ) . By (b), f^{-1} is Q^* -continuous and so $(f^{-1})^{-1}(F) = f(F)$ is Q^* -closed in (Y, σ) . This proves (c).

Step 3: (c) \Rightarrow (a)

Follows from Proposition 3.2.

Definition 3.7 Let A be a subset of X . A mapping $r: X \rightarrow A$ is called a Q^* -continuous retraction if r is Q^* -continuous and the restriction τ_A is the identity mapping on A .

Definition 3.8 A topological space (X, τ) is called a Q^* -Hausdorff if for each pair x, y of distinct points of X , there exists Q^* -neighborhoods U_1 and U_2 of x and y , respectively, that are disjoint.

Lemma 3.1 Let A be a subset of X . Then $p \in Q^* - \text{cl}(A)$ if and only if for any Q^* -neighborhood N of p in X , $A \cap N \neq \emptyset$

Theorem 3.1 Let A be a subset of X and $r: X \rightarrow A$ be a Q^* -continuous retraction. If X is Q^* -Hausdorff, then A is a Q^* -closed set of X .

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Proof. Suppose that A is not Q^* -closed. Then there exists a point x in X such that $x \in Q^* - \text{cl}(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because r is Q^* -continuous retraction. Since X is Q^* -Hausdorff, there exists disjoint Q^* -open sets U and V in X such that $x \in U$ and $r(x) \in V$. Now let W be an arbitrary Q^* -neighborhood of x . Then $W \cap U$ is a Q^* -neighborhood of x .

Since $x \in Q^* - \text{cl}(A)$, by Lemma 3.1, we have $(W \cap U) \cap A \neq \emptyset$. Therefore there exists a point y in $W \cap U \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \notin V$. This implies that $r(W) \not\subset V$ because $y \in W$. This is contrary to the Q^* -continuity of r . Consequently, A is a Q^* -closed set of X . ■

Theorem 3.1 Let $f: X \rightarrow Y$ be a bijection. If f is pre Q^* -open then $f^{-1}: X \rightarrow Y$ is irresolute.

Proof. Suppose that the bijection f is pre Q^* -open then f^{-1} is a function from Y to X . Let U be a Q^* -open set in X . Since every Q^* -open set is semi open we have U is semi open in X . Then $f(U)$ is a semi open set in Y . But $f(U) = (f^{-1})^{-1}(U)$. Hence $(f^{-1})^{-1}(U)$ is a semi open set in Y . Put $g = f^{-1}$. We have $g^{-1}(U)$ is semi open. Consequently, g is irresolute. That is, f^{-1} is irresolute. ■

Theorem 3.2 Let $f: X \rightarrow Y$ be a bijection. Then f is pre Q^* -open if and only if $f^{-1}: Y \rightarrow X$ is Q^* -irresolute.

Proof.

Step 1 : Suppose that the bijection f is pre Q^* -open then f^{-1} is a function from Y to X . Let U be a Q^* -open set in X . Then $f(U)$ is a Q^* -open set in Y . But $f(U) = (f^{-1})^{-1}(U)$. Hence $(f^{-1})^{-1}(U)$ is a Q^* -open set in Y . Put $g = f^{-1}$. We have $g^{-1}(U)$ is Q^* -open. Consequently, g is Q^* -irresolute. That is, f^{-1} is Q^* -irresolute.

Step 2 : Suppose that f is Q^* -irresolute. Let U be a Q^* -open set in X . Then $(f^{-1})^{-1}(U)$ is a Q^* -open set in Y . But $(f^{-1})^{-1}(U) = f(U)$. Therefore $f(U)$ is a Q^* -open set in Y . Hence f is pre Q^* -open. ■

Theorem 3.3 If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both Q^* -irresolute, then their composition $g \circ f: X \rightarrow Z$ is an Q^* -irresolute map.

Proof. Let V be Q^* -open set in Z . Then $(g \circ f)^{-1}(V) = (f^{-1} \circ g^{-1})(V) = (f^{-1}(g^{-1}(V)))$. Since g is Q^* -irresolute, it follows that $g^{-1}(V)$ is a Q^* -open set. Since f is Q^* -irresolute, it follows that $f^{-1}(g^{-1}(V))$ is a Q^* -open set. Thus for each Q^* -open set V in Z , $(g \circ f)^{-1}(V)$ is Q^* -open in X . Therefore, $g \circ f$ is an Q^* -irresolute function. ■

Theorem 3.4 Q^* -homeomorphism is an equivalence relation. We write $X \sim Y$ whenever two spaces X, Y are Q^* -homeomorphic.

Proof.

Step 1 : Let $i: X \rightarrow X$ be the identity map on X . Then it is bijective and Q^* -irresolute. Also $(i)^{-1}$ is a pre Q^* -open map. Hence i is a Q^* -homeomorphism. Accordingly $X \sim X$. The relation is reflexive.

Step 2 : Suppose that $X \sim Y$. Then there exists a Q^* -homeomorphism $h: X \rightarrow Y$. But then h is bijective. Accordingly $h^{-1}: Y \rightarrow X$ is bijective. Also h is Q^* -irresolute. Hence h^{-1} is a pre Q^* -open map. Hence $Y \sim X$.

Step 3: Suppose that $X \sim Y$ and $Y \sim Z$. Then there is a Q^* -homeomorphism $f: X \rightarrow Y$ and there is a Q^* -homeomorphism g from Y to Z . But then f and g are bijective. Accordingly, $g \circ f$ is bijective and pre Q^* -open. Thus, $g \circ f$ is Q^* -homeomorphism. Therefore, $X \sim Z$. Hence \sim is transitive. From step (1), (2) and (3), Q^* -homeomorphism is an equivalence relation. ■

Theorem 3.5 Every Q^* -compact subset of a Hausdorff space is semiclosed.

Proof. Suppose that A be a Q^* -compact subset of a Hausdorff space X . Since every Q^* -compact space is semi compact we have A is semi compact. Let $x \in X - A$. Then there exists 2 disjoint semi open sets U_x and V_x such that $x \in U_x$ and

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$A \subset V_x$. But then $X \in U_x \subset X - V_x \subset X - A$. Therefore, $X - A$ is semi open. Hence A is semi - closed in Y . ■

Theorem 3.6 Every Q^* - compact subset of a Q^* - Hausdorff space is Q^* -closed.

Proof. Suppose that A be a Q^* - compact subset of a Q^* - Hausdorff space X . Let $x \in X - A$. Then there are 2 disjoint Q^* - open sets U_x and V_x such that $x \in U_x$ and $A \subset V_x$. But then $X \in U_x \subset X - V_x \subset X - A$. Therefore, $X - A$ is Q^* - open. Hence A is Q^* - closed in Y . ■

Theorem 3.7 Let X be Q^* compact and set Y be a Hausdorff space. If $f : X \rightarrow Y$ is Q^* - continuous, Q^* - irresolute and bijective, then f is a Q^* -homeomorphism.

Proof. Let A be a Q^* - closed subset of the Q^* - compact space X . Then A is Q^* -compact. But f is Q^* - irresolute. Hence $f(A)$ is Q^* - compact. Take $g = f^{-1}$. Then $g^{-1}(A)$ is Q^* -closed, by theorem 3.4. Consequently g is an Q^* - irresolute map. That is, f^{-1} is Q^* - irresolute. Therefore, f is a Q^* - homeomorphism. ■

Proposition 3.4 If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \gamma)$ are Q^* - homeomorphisms, then their composition $g \circ f : (X, \tau) \rightarrow (Z, \gamma)$ is also Q^* - homeomorphism.

Proof. Let U be a Q^* - open set in (Z, γ) . Now, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, V is Q^* - open in (Y, σ) and so again by hypothesis, $f^{-1}(V)$ is Q^* - open in (X, τ) . Therefore, $g \circ f$ is Q^* - irresolute. Also for a Q^* - open set G in (X, τ) , we have $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$. By hypothesis $f(G)$ is Q^* - open in (Y, σ) and so again by hypothesis, $g(f(G))$ is Q^* - open in (Z, γ) . i.e., $(g \circ f)(G)$ is Q^* - open in (Z, γ) and therefore $(g \circ f)^{-1}$ is Q^* - irresolute. Hence $g \circ f$ is a Q^* -homeomorphism. ■

Example 3.3 Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{\emptyset, Y, \{b\}, \{b, c\}\}$, $\eta = \{\emptyset, Z, \{b, c\}\}$. Then $\emptyset, \{a\}, \{a, c\}$ are Q^* - closed in Y . Let $f : X \rightarrow Y$ be the identity map. Then f and g are Q^* - homeomorphism. Here $g \circ f$ is Q^* - continuous, since $\{b, c\}$ is Q^* - open in Z and $(g \circ f)^{-1}(\{b, c\}) = \{b, c\}$ is Q^* - open in X . Hence $g \circ f$ is Q^* homeomorphism.

Remark 3.1. We denote the family of all Q^* homeomorphisms from (X, τ) onto itself by $Q^* - h(X, \tau)$.

Theorem 3.8 The set $Q^* - h(X, \tau)$ is a group under the composition of maps.

Proof. Define a binary operation $*$: $Q^* - h(X, \tau) \times Q^* - h(X, \tau) \rightarrow Q^* - h(X, \tau)$ by $f * g = g \circ f$ for all $f, g \in Q^* - h(X, \tau)$ and \circ is the usual operation of composition of maps. Then by Proposition 3.4, $g \circ f \in Q^* - h(X, \tau)$. We know that the composition of maps is associative and the identity map $I : (X, \tau) \rightarrow (X, \tau)$ belonging to $Q^* - h(X, \tau)$ serves as the identity element. If $f \in Q^* - h(X, \tau)$, then $f^{-1} \in Q^* - h(X, \tau)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of $Q^* - h(X, \tau)$. Therefore, $(Q^* - h(X, \tau), \circ)$ is a group under the operation of composition of maps. ■

Theorem 3.9 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. Then f induces an isomorphism from the group $Q^* - h(X, \tau)$ onto the group $Q^* - h(Y, \sigma)$.

Proof. Using the map f , we define a map $\theta_f : Q^* - h(X, \tau) \rightarrow Q^* - h(Y, \sigma)$ by $\theta_f(h) = f \circ h \circ f^{-1}$ for every $h \in Q^* - h(X, \tau)$. Then θ_f is a bijection. Further, for all $h_1, h_2 \in Q^* - h(X, \tau)$, $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, θ_f is a homeomorphism and so it is an isomorphism induced by f . ■

Proposition 3.4 For any two subsets A and B of (X, τ) ,

- (i) If $A \subset B$, then $Q^* - cl(A) \subset Q^* - cl(B)$,
- (ii) $Q^* - cl(A \cap B) \subset Q^* - cl(A) \cap Q^* - cl(B)$. ■

Theorem 3.10. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then $Q^* - cl(f^{-1}(B)) = f^{-1}(Q^* - cl(B))$ for all $B \subset Y$.

Proof. Since f is a Q^* -homeomorphism, f is Q^* - irresolute. Since $Q^* - cl(f(B))$ is a Q^* -closed set in (Y, σ) , $f^{-1}(Q^* - cl(f(B)))$ is Q^* -closed in (X, τ) . Now, $f^{-1}(B) \subset f^{-1}(Q^* - cl(f(B)))$ and so by Proposition 3.4, $Q^* - cl(f^{-1}(B)) \subset f^{-1}(Q^* - cl(f(B)))$. Again since f is a Q^* -homeomorphism, f^{-1} is Q^* - irresolute. Since $Q^* - cl(f^{-1}(B))$ is Q^* -closed in (X, τ) , $(f^{-1})^{-1}(Q^* - cl(f^{-1}(B))) = f(Q^* - cl(f^{-1}(B)))$ is Q^* -closed in (Y, σ) . Now, $B \subset (f^{-1})^{-1}(f^{-1}(B)) \subset (f^{-1})^{-1}(Q^* - cl(f^{-1}(B))) = f(Q^* - cl(f^{-1}(B)))$ and so

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$Q^*-cl(B) \subset f(Q^*-cl(f^{-1}(B)))$. Therefore, $f^{-1}(Q^*-cl(B)) \subset f^{-1}(f(Q^*-cl(f^{-1}(B)))) \subset Q^*-cl(f^{-1}(B))$ and hence the equality holds. ■

Corollary 3.1 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then $Q^*-cl(f(B)) = f(Q^*-cl(B))$ for all $B \subset X$.

Proof. Since $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, $f^{-1}: (Y, \sigma) \rightarrow (X, \tau)$ is also a Q^* -homeomorphism. Therefore, by theorem 3.10, $Q^*-cl((f^{-1})^{-1}(B)) = (f^{-1})^{-1}(Q^*-cl(B))$ for all $B \subset X$. i.e., $Q^*-cl(f(B)) = f(Q^*-cl(B))$. ■

Corollary 3.2 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then $f(Q^*-int(B)) = Q^*-int(f(B))$ for all $B \subset X$.

Proof. For any set $B \subset X$, $Q^*-int(B) = (Q^*-cl(B^c))^c$. Thus, by utilizing corollary 3.2, we obtain $f(Q^*-int(B)) = f((Q^*-cl(B^c))^c) = (f(Q^*-cl(B^c)))^c = (Q^*-cl(f(B^c)))^c = (Q^*-cl((f(B))^c))^c = Q^*-int(f(B))$. ■

Corollary 3.3 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then $f^{-1}(Q^*-int(B)) = Q^*-int(f^{-1}(B))$ for all $B \subset Y$.

Proof. Since $f^{-1}: (Y, \sigma) \rightarrow (X, \tau)$ is also a Q^* -homeomorphism, the proof follows from Corollary 3.2. ■

IV. Q^* - HOMEOMORPHISM

Throughout this paper X and Y always represent nonempty topological spaces (X, τ) and (Y, σ) . The family of all closed set in (X, τ) is denoted by $C(X)$.

Theorem 4.1 Every Q^* homeomorphism is a homeomorphism.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. Then f is bijective and both f and f^{-1} are Q^* -continuous. Since every Q^* continuous function is continuous we have f and f^{-1} are continuous. This shows that f is a homeomorphism. ■

Remark 4.1 The converse of the above theorem need not be true, as shown in the following example.

Example 4.1 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{a\}, \{b, c\}\}$ and $\sigma = \{\phi, Y, \{b\}, \{b, c\}\}$. Let $f: X \rightarrow Y$ be the identity map. Therefore, f is a homomorphism but not Q^* homeomorphism.

Definition 4.1 For a subset A of a space (X, τ_1, τ_2) we define the Q^* kernel of A (briefly, $Q^* \ker(A)$) as follows: $Q^* \ker(A) = \bigcap \{F : F \in Q^*O(X); A \subset F\}$. A is said to be a Q^* - Λ set in X if $A = Q^* \ker(A)$, or equivalently, if A is the intersection of Q^* open sets. A is said to be Q^* - λ -closed in X if it is the intersection of a Q^* - Λ set in X and clearly, Q^* - Λ sets and Q^* closed sets are Q^* - λ closed; complements of Q^* - λ closed sets in X are said to be Q^* - λ open in X .

Proposition 4.1 For a subset A of a space X , the following are equivalent:

- (i) A is Q^* - λ closed in X .
- (ii) $A = L \cap Q^*-cl(A)$, where L is a Q^* - Λ set in X .
- (iii) $A = Q^*-\ker(A) \cap Q^*-cl(A)$. ■

Definition 4.2 A bijection $f: X \rightarrow Y$ is called Q^* -homeomorphism, if f is Q^* irresolute and its inverse also Q^* irresolute.

Remark 4.2 We say that spaces X and Y are Q^* -homeomorphic if there exists a Q^* homeomorphism from (X, τ) onto (Y, σ) . We denote the family of all Q^* homeomorphisms from (X, τ) onto itself by $Q^*-H(X)$.

Theorem 4.2 Every Q^* -homeomorphism is a Q^* -homeomorphism.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. Then f is bijective, Q^* irresolute and f^{-1} is Q^* irresolute. Since every Q^* irresolute is Q^* continuous, f and f^{-1} are Q^* continuous and so f is a Q^* homeomorphism. ■

Remark 4.3 The following example shows that the converse of the above theorem need not be true.

Example 4.3 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}, \{b\}\}$ and $\sigma = \{\phi, Y, \{b\}, \{c\}\}$. Let $f: X \rightarrow Y$ be a map defined by $f(a) = f(b) = c$ and $f(c) = b$. Therefore, f is a Q^* -homomorphism but not Q^* -homeomorphism.

Remark 4.4 The concepts of homeomorphisms and Q^* -homeomorphism are independent.

Example 4.4 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}\}$. Let $f: X \rightarrow Y$ be a identity map. Then f is homeomorphism but not Q^* homeomorphism.

Lemma 4.1 A function $f: X \rightarrow Y$ is Q^* -irresolute if and only if $f^{-1}(V)$ is a Q^* -closed set in X for every Q^* -closed set V in Y .

Theorem 4.3 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then $Q^*-cl(f^{-1}(B)) = f^{-1}(Q^*-cl(B))$ for every $B \subseteq Y$.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. Then by Definition 4.2, both f and f^{-1} are Q^* -irresolute and f is bijective. Let $B \subseteq Y$. Since $Q^*-cl(B)$ is a Q^* -closed set in (Y, σ) , using Lemma 4.1, $f^{-1}(Q^*-cl(B))$ is Q^* -closed in (X, τ) . But

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$Q^*-cl(f^{-1}(B))$ is the smallest Q^* -closed set containing $f^{-1}(B)$.

Therefore $Q^*-cl(f^{-1}(B)) \subseteq f^{-1}(Q^*-cl(B))$. $\rightarrow (1)$.

Again, $Q^*-cl(f^{-1}(B))$ is Q^* -closed in (X, τ) . Since f^{-1} is Q^* -irresolute, $f(Q^*-cl(f^{-1}(B)))$ is Q^* -closed in (Y, σ) . Now, $B = f(f^{-1}(B)) \subseteq f(Q^*-cl(f^{-1}(B)))$. Since $f(Q^*-cl(f^{-1}(B)))$ is Q^* -closed and $Q^*-cl(B)$ is the smallest Q^* -closed set containing B , $Q^*-cl(B) \subseteq f(Q^*-cl(f^{-1}(B)))$ that implies $f^{-1}(Q^*-cl(B)) \subseteq f^{-1}(f(Q^*-cl(f^{-1}(B)))) = Q^*-cl(f^{-1}(B))$.

That is, $f^{-1}(Q^*-cl(B)) \subseteq Q^*-cl(f^{-1}(B)) \rightarrow (2)$

From (1) and (2), $Q^*-cl(f^{-1}(B)) = f^{-1}(Q^*-cl(B))$. ■

Corollary 4.1 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then $Q^*-cl(f(B)) = f(Q^*-cl(B))$ for every $B \subseteq X$.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. Since f is Q^* -homeomorphism, f^{-1} is also a Q^* -homeomorphism. Therefore by Theorem 4.3, it follows that $Q^*-cl(f(B)) = f(Q^*-cl(B))$ for every $B \subseteq X$. ■

Corollary 4.2 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then $f(Q^*-int(B)) = Q^*-int(f(B))$ for every $B \subseteq X$.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. For any set $B \subseteq X$, $Q^*-int(B) = (Q^*-cl(B^c))^c$. $f(Q^*-int(B)) = f((Q^*-cl(B^c))^c) = (f(Q^*-cl(B^c)))^c = (f(Q^*-cl(B^c)))^c$. Then using Corollary 4.2, we see that $f(Q^*-int(B)) = (Q^*-cl(f(B^c)))^c = Q^*-int(f(B))$. ■

Corollary 4.3 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then for every $B \subseteq Y$, $f^{-1}(Q^*-int(B)) = Q^*-int(f^{-1}(B))$.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. Since f is Q^* -homeomorphism, f^{-1} is also a Q^* -homeomorphism. Therefore by Corollary 4.2, $f^{-1}(Q^*-int(B)) = Q^*-int(f^{-1}(B))$ for every $B \subseteq Y$. ■

Theorem 4.3 If $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \gamma)$ are Q^* -homeomorphisms, then their composition $g \circ f: (X, \tau) \rightarrow (Z, \gamma)$ is also Q^* -homeomorphism.

Proof. Let U be a Q^* -open set in (Z, γ) . Since g is Q^* -homeomorphism, g is Q^* -irresolute and so $g^{-1}(U)$ is Q^* -open in (Y, σ) . Since f is Q^* -homeomorphism, f is Q^* -irresolute and so $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is Q^* -open in (X, τ) . This implies that $g \circ f$ is Q^* -irresolute. Again let G be Q^* -open in (X, τ) . Since f is Q^* -homeomorphism, f^{-1} is Q^* -irresolute and so $(f^{-1})^{-1}(G) = f(G)$ is Q^* -open in (Y, σ) . Since g is Q^* -homeomorphism, g^{-1} is Q^* -irresolute and so $(g^{-1})^{-1}(f(G)) = g(f(G)) = (g \circ f)(G) = ((g \circ f)^{-1})^{-1}(G)$ is Q^* -open in (Z, γ) . This implies that $(g \circ f)^{-1}$ is Q^* -irresolute. Since f and g are Q^* -homeomorphism f and g are bijective and so $g \circ f$ is bijective. This completes the proof. ■

Theorem 4.4 The set $Q^*H(X, \tau)$ is a group under composition of functions.

Proof. Let $f, g \in Q^*H(X, \tau)$. Then $f \circ g \in Q^*H(X, \tau)$ by Theorem 3.17. Since f is bijective, $f^{-1} \in Q^*H(X, \tau)$. This completes the proof. ■

Theorem 4.5 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a Q^* -homeomorphism, then f induces an isomorphism from the group $Q^*H(X, \tau)$ onto the group $Q^*H(Y, \sigma)$.

Proof. Let $f \in Q^*H(X, \tau)$. Then define a map $\psi_f: Q^*H(X, \tau) \rightarrow Q^*H(Y, \sigma)$ by $\psi_f(h) = f \circ h \circ f^{-1}$ for every $h \in Q^*H(X, \tau)$. Let $h_1, h_2 \in Q^*H(X, \tau)$.

$$\begin{aligned} \text{Then } \psi_f(h_1 \circ h_2) &= f \circ (h_1 \circ h_2) \circ f^{-1} \\ &= f \circ (h_1 \circ f^{-1} \circ f \circ h_2) \circ f^{-1} \\ &= (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) \\ &= \psi_f(h_1) \circ \psi_f(h_2). \end{aligned}$$

Since $\psi_f(f^{-1} \circ h \circ f) = h_1 \psi_f$ is onto. Now, $\psi_f(h) = I$ implies $f \circ h \circ f^{-1} = I$. That implies $h = I$. This proves that ψ_f is one-one. This shows that ψ_f is an isomorphism. ■

Theorem 4.6 Every Q^* -open set is Λ_r -open.

Theorem 4.7 Every Q^* -homeomorphism is a Λ_r -homeomorphism.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a Q^* -homeomorphism. Then f is bijective and both f and f^{-1} are Q^* -continuous. Since every Q^* -continuous function is Λ_r -continuous, f and f^{-1} are Λ_r -continuous. This shows that f is a Λ_r -homeomorphism.

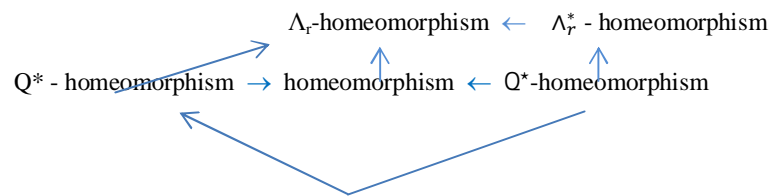
Remark 4.5 The converse of the above theorem need not be true, as shown in the following example.

Example 4.4 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then $\Lambda_r O(X, \tau) = \tau$ and $\Lambda_r O(Y, \sigma) = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then f is Λ_r -homeomorphism. Since $f(\{b, c\}) = \{a, b\}$ is not Q^* -open in (Y, σ) , f^{-1} is not Q^* -continuous that implies f is not a Q^* -homeomorphism.

V. CONCLUSION

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In this paper, we introduce two classes of maps called Q^* - homeomorphisms and Q^* -homeomorphisms and study their properties. From all of the above statements, we have the following diagram:



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