# The Non-Homogeneous Quintic Equation with Six Unknowns $x^{4}-y^{4}=109(z+w) P^{3} Q$ 

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Abstract: The non-homogeneous quintic equation with six unknowns given by $x^{4}-y^{4}=109(z+w) P^{3} Q$ is analyzed for its patterns of non-zero distinct integer solutions.
Keywords: Non - homogeneous Quintic, Quintic with six unknowns, Diophantine equations, Integral solutions, Special numbers.
Notations

Special numbers
Regular Polygonal Number
Pronic Number
Pyramidal number

Notations

$$
\mathrm{t}_{\mathrm{m}, \mathrm{n}}
$$

$$
\mathrm{Pr}_{\mathrm{n}}
$$

$$
P_{n}^{m}
$$

## I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, Quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-2]. For illustration, one may refer [3-5] for Quintic equation with three unknowns and [6-8] for Quintic equation with five unknowns. This paper concerns with the problem of the non-homogeneous Quintic equation with six unknowns given by $x^{4}-y^{4}=109(z+w) P^{3} Q$. A few relations among the solutions are presented.

## II. METHOD OF ANALYSIS

The non-homogeneous quintic equation with six unknowns to be solved for its distinct non-zero integral solution is
$x^{4}-y^{4}=109(z+w) P^{3} Q$
Assume
$x=u+v, y=u-v, z=2 u+v, w=2 u-v, Q=2 v$
Substituting (2) in (1), it leads to
$u^{2}+v^{2}=109 P^{3}$
Different methods of solutions of the above equation are given below.

1) Method 1

Assume $P=a^{2}+b^{2}$
where a and b are non-zero distinct integers.
Write 109 as $109=(10+i 3)(10-i 3)$
Substituting (4) and (5) in (3) and applying the method of factorization, define
$(u+i v)(u-i v)=(10+i 3)(10-i 3)(a+i b)^{3}(a-i b)^{3}$
Equating positive and negative factors, we get
$u+i v=(10+i 3)(a+i b)^{3}$
$u-i v=(10-i 3)(a-i b)^{3}$
Equating the real and imaginary parts in either of the above two equations, we get
$u=10 a^{3}-9 a^{2} b-30 a b^{2}+3 b^{3}$
$v=3 a^{3}+30 a^{2} b-9 a b^{2}-10 b^{3}$
Hence, in view of (2) and (4), we have
$x=x(a, b)=13 a^{3}+21 a^{2} b-39 a b^{2}-7 b^{3}$
$y=y(a, b)=7 a^{3}-39 a^{2} b-21 a b^{2}+13 b^{3}$
$z=z(a, b)=23 a^{3}+12 a^{2} b-69 a b^{2}-4 b^{3}$
$w=w(a, b)=17 a^{3}-48 a^{2} b-51 a b^{2}+16 b^{3}$
$Q=Q(a, b)=6 a^{3}+60 a^{2} b-18 a b^{2}-20 b^{3}$
$P=P(a, b)=a^{2}+b^{2}$
which satisfy (1).
a) Properties

1) $x^{3}-3 x y Q=y^{3}+Q^{3}$
2) $x(a, 1)-26 P_{5}^{a}-16 t_{3, a-1}+7 \equiv 0(\bmod 31)$
3) $x(a, 1)-26 P_{5}^{a}-t_{18, a}+7 \equiv 0(\bmod 32)$
4) $x(a, 1)-26 P_{5}^{a}-16 t_{3, a}+7 \equiv 0(\bmod 47)$

Note
In addition to (5), 109 may also be represented as
$109=(3+i 10)(3-i 10)$
Proceeding as in method 1 , another set of solutions to (1) is exhibited below:

$$
\begin{aligned}
& x=x(a, b)=13 a^{3}-21 a^{2} b-39 a b^{2}+7 b^{3} \\
& y=y(a, b)=-7 a^{3}-39 a^{2} b+21 a b^{2}+13 b^{3} \\
& z=z(a, b)=16 a^{3}-51 a^{2} b-48 a b^{2}+17 b^{3} \\
& w=w(a, b)=-4 a^{3}-69 a^{2} b+12 a b^{2}+23 b^{3} \\
& Q=Q(a, b)=20 a^{3}+18 a^{2} b-60 a b^{2}-6 b^{3} \\
& P=P(a, b)=a^{2}+b^{2}
\end{aligned}
$$

2) Method 2

Rewrite (3) as

$$
\begin{equation*}
u^{2}+v^{2}=109 P^{3} * 1 \tag{6}
\end{equation*}
$$

Assume $\quad 1=\frac{(3+4 i)(3-4 i)}{25}$
Following the analysis presented in method- 1 ,

$$
\begin{align*}
& u=\frac{1}{5}\left[18 a^{3}-147 a^{2} b-54 a b^{2}+49 b^{3}\right]  \tag{*}\\
& v=\frac{1}{5}\left[49 a^{3}+54 a^{2} b-147 a b^{2}-18 b^{3}\right]
\end{align*}
$$

Since our interest is on finding integer solutions, replacing a by 5A and b by 5B in (*), (4) and using (2), the corresponding integer solutions to (1) are found to be
$x=x(A, B)=1675 A^{3}-2325 A^{2} B-5025 A B^{2}+775 B^{3}$
$y=y(A, B)=-775 A^{3}-5025 A^{2} B+2325 A B^{2}+1675 B^{3}$
$z=z(A, B)=2125 A^{3}-6000 A^{2} B-6375 A B^{2}+2000 B^{3}$
$w=w(A, B)=-325 A^{3}-8700 A^{2} B+975 A B^{2}+2900 B^{3}$
$Q=Q(A, B)=2450 A^{3}+2700 A^{2} B-7350 A B^{2}-900 B^{3}$
$P=P(A, B)=25\left(A^{2}+B^{2}\right)$
a) Properties

1) $Q(A, 1)-4900 p_{5}^{A}-500 t_{3, A}+$ square number $\equiv 0(\bmod 7600)$
2) $y(1, B)-3350 P_{5}^{B}-t_{1300, B}+775 \equiv 0(\bmod 4376)$
3) $4\left[z(A, 1)-4250 P_{5}^{A}+6375 \operatorname{Pr}_{A}+1750 t_{4, A}\right]$ is a cubical integer.
4) $w(1, B)-5800 P_{5}^{B}+1925 t_{4, A}+325 \equiv 0(\bmod 8700)$

Note: In addition to (7), 1 may also be represented by $1=\frac{(1+i)^{2 n}(1-i)^{2 n}}{2^{2 n}}$
For this choice, the corresponding integer solutions to (1) are found to be

$$
\begin{aligned}
x=x(a, b)= & \cos n \pi / 2\left[13 a^{3}-39 a b^{2}-21 a^{2} b+7 b^{3}\right] \\
& -\sin n \pi / 2\left[7 a^{3}-21 a b^{2}+39 a^{2} b-13 b^{3}\right] \\
y=y(a, b)= & \cos n \pi / 2\left[-7 a^{3}+21 a b^{2}-39 a^{2} b+13 b^{3}\right] \\
& -\sin n \pi / 2\left[13 a^{3}-39 a b^{2}-21 a^{2} b+7 b^{3}\right] \\
z=z(a, b)= & \cos n \pi / 2\left[16 a^{3}-48 a b^{2}-51 a^{2} b+17 b^{3}\right] \\
& -\sin n \pi / 2\left[17 a^{3}-51 a b^{2}+48 a^{2} b-16 b^{3}\right] \\
w=w(a, b)= & \cos n \pi / 2\left[-4 a^{3}+12 a b^{2}-69 a^{2} b+23 b^{3}\right] \\
& -\sin n \pi / 2\left[23 a^{3}-69 a b^{2}-12 a^{2} b+4 b^{3}\right] \\
Q=Q(a, b)= & \cos n \pi / 2\left[20 a^{3}-60 a b^{2}+18 a^{2} b-6 b^{3}\right] \\
& -\sin n \pi / 2\left[6 a^{3}-18 a b^{2}-60 a^{2} b+20 b^{3}\right] \\
P= & P(a, b)=
\end{aligned} a^{2}+b^{2} \quad l
$$

## 3) Method 3

Taking

$$
\begin{equation*}
u=109^{2} U, v=109^{2} V \& P=109 R \tag{8}
\end{equation*}
$$

in (3), it becomes $\quad U^{2}+V^{2}=R^{3}$
which is satisfied by
$U=m\left(m^{2}+n^{2}\right)$
$V=n\left(m^{2}+n^{2}\right)$
$R=\left(m^{2}+n^{2}\right)$
Substituting (10) in (8) and using (2), the integer solutions to (1) are given by
$x=109^{2}(m+n)\left(m^{2}+n^{2}\right)$
$y=109^{2}(m-n)\left(m^{2}+n^{2}\right)$
$z=109^{2}(2 m+n)\left(m^{2}+n^{2}\right)$
$w=109^{2}(2 m-n)\left(m^{2}+n^{2}\right)$
$q=2\left(109^{2}\right)\left[n\left(m^{2}+n^{2}\right)\right]$
$p=109\left[m^{2}+n^{2}\right]$
a) Properties

1) $2[z(m, n)-x(m, n)-y(m, n)]-Q(m, n)=0$
2) $z(m, n)-w(m, n)-Q(m, n)=0$
3) $x(m, n)+y(m, n)-z(m, n)+109 P=0$
4) $x(m, n)+y(m, n)-z(m, n)+w(m, n)-Q(m, n) \equiv 0(\bmod 109)$

Note: It is to be noted that (9) is also satisfied by

$$
\begin{aligned}
U & =\left(m^{3}-3 m n^{2}\right) \\
V & =\left(3 n m^{2}-n^{3}\right) \\
R & =\left(m^{2}+n^{2}\right)
\end{aligned}
$$

In this case, the integer solutions to (1) are seen to be

$$
\begin{aligned}
& x=109^{2}\left[m^{3}+3 m^{2} n-3 m n^{2}-n^{3}\right] \\
& y=109^{2}\left[m^{3}-3 m^{2} n-3 m n^{2}+n^{3}\right] \\
& z=109^{2}\left[2 m^{3}+3 m^{2} n-6 m n^{2}-n^{3}\right] \\
& w=109^{2}\left[2 m^{3}-3 m^{2} n-6 m n^{2}+n^{3}\right] \\
& Q=2\left[109^{2}\right]\left[3 n m^{2}-n^{3}\right] \\
& P=109\left[m^{2}+n^{2}\right]
\end{aligned}
$$

## III. CONCLUSION

In this paper, we have illustrated different methods of obtaining non-zero integer solutions to the quintic equation with six unknowns given by $x^{4}-y^{4}=109(z+w) P^{3} Q$. As the quintic Diophantine equation are rich in variety one may consider other forms of quintic equation with variable $\geq 6$ and search for their corresponding integer solutions along with the corresponding properties.

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