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Numerical Results of Sub-Cauchy Problem for Linear Elasticity

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Abstract: This work focuses on the sub-Cauchy problem for linear elasticity in two dimentional case. Solving such a problem may be formulated as follows: given the displacement and one component of the traction in a given part of the boundary of the elastic body, reconstruct the displacement field in all the domain. Author propose herein, an iterative method borrowed from the domain decomposition communauty to solve the sub-Cauchy problem. Numerical results highlight the efficiency of the proposed method.
Keywords: Linear elasticity; Shear stress; Cauchy problem; Steklov Poincare operator; Domain decomposition; Inverse problems

I. INTRODUCTION

Many inverse problems in linear elasticity are defined by overdetermined boundary conditions. One can think to the reconstruction of buried flaws such as cracks, voids or inhomogeneities, identification of constitutive law, data completion (that is the recovery of boundary conditions on an inaccessible part of the body boundary) [1].All the above inverse problems have in common to be defined by overspecified boundary conditions namely the normal stress and the displacement on a part of the boundary which correspond to Cauchy data. Many papers treated this problem, from the numerical view point, this last decade [2-4].

Author would like to mention the work by Bourgeois [5] who applied the Lions-quasi-reversibility method to the data completion. This method leads to a direct inversion process.

Many authors resort to iterative methods based on minimising a least-square type error functional, [6-8]. Marin [9] would like to mention the minimization of an energy-like gap functional in ref. [10] and domain decomposition like method in ref. [11] which are close to what we develop in this work.

Hereafter, Author are concerned by a partially overdetermined boundary conditions. In fact,on a part of the boundary of the domain partially overdetermined boundary data are prescribed, namely one component of the traction and the displacement field. Following ref. [12] author build an energy-gap error functional to recover the lacking boundary data. Author emphasise on the shear stress reconstruction, on the part of boundary where the partial-data is prescribed.

II. FORMULATION OF SUB-CAUCHY PROBLEM AS STEKLOV POINCARE OPERATOR

The inverse problem under consideration concerns the recovery of lacking boundary data from the knowledge of partially overdetermined boundary elastic data. The problem is formulated mathematically as follows : Let Ω be a bounded domain in R^2, the boundary $\Gamma=\partial\Omega$ is split into $\Gamma_c$and Γ_i having both non vanishing measure $\Gamma_c \cap \Gamma_i = \emptyset$. Given the displacement U and the normal component of surface traction $\Phi.n$ on Γ_c:

$$\begin{cases} div\,\sigma(u) = 0 & in \quad \Omega, \\ (\sigma(u)\,n).n = \Phi.n & on \quad \Gamma_c, \\ u = U & on \quad \Gamma_c. \end{cases} \tag{1}$$

where $\sigma = \lambda Tr\varepsilon(u)+2\mu\varepsilon\ (u)$, $\varepsilon = 1/2(\nabla u+\nabla u_T)$ and λ, μ are the Lamé coefficients related to Young's modulus E and the Poisson ratio v via:

$$\mu = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$$

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Our aim is then to reconstruct (σ (u).n).τ on Γ_c and both the displacement and traction. To our knowledge, there are no theoretical studies (existence and uniqueness) of this problem despite its great importance in applications. In this paper author treat this problem numerically by solving a data completion problem.

The decomposition of the **Cauchy problem** (1) is formulated through an unknown function η as follows:

$$(P_D)\begin{cases} div\,\sigma(u_D) = 0 & in \quad \Omega, \\ u_D = U & on \quad \Gamma_c, \\ u_D = \eta & on \quad \Gamma_i. \end{cases} \quad (P_N)\begin{cases} div\,\sigma(u_N) = 0 & in \quad \Omega, \\ (\sigma(u_N).n).n = \Phi n & on \quad \Gamma_c, \\ u_N.\tau = U.\tau & on \quad \Gamma_c, \\ u_N = \eta & on \quad \Gamma_i. \end{cases} \tag{2}$$

Where η is the virtual control and u is chosen so that u_D and u_N adjust in the best possible on Ω. The solution u_D and u_N are a function of η ($u_D = u_D(\eta)$ and $u_N = u_N(\eta)$).

To express the problem in the framework of virtual control, we introduce the cost functional:

$$J(\eta) = \int_\Omega \sigma(u_D - u_N) : \varepsilon(u_D - u_N) \quad d\Omega \tag{3}$$

and consider the minimization problem:

$$\inf_{\eta \in H^{\frac{1}{2}}(\Gamma_i)} J(\eta) \tag{4}$$

The solutions u_D and u_N can be written as:

$$u_D = u_D^0 + u_D^* \qquad u_N = u_N^0 + u_N^*$$

Where u_i^0 depends on the data U and Φ.n where as u_i^* depends on η as follows:

$$(P_D^*)\begin{cases} div\,\sigma(u_D^*) = 0 & in \quad \Omega, \\ u_D^* = 0 & on \quad \Gamma_c, \\ u_D^* = \eta & on \quad \Gamma_i. \end{cases} \quad (P_D^0)\begin{cases} div\,\sigma(u_D^0) = 0 & in \quad \Omega, \\ u_D^0 = U & on \quad \Gamma_c, \\ u_D^0 = 0 & on \quad \Gamma_i. \end{cases} \tag{5}$$

Similary, author decompose u_N.

$$(P_N^*)\begin{cases} div\,\sigma(u_N^*) = 0 & in \quad \Omega, \\ (\sigma(u_N).n).n = 0 & on \quad \Gamma_c, \\ u_N.\tau = 0 & on \quad \Gamma_c, \\ u_N^* = \eta & on \quad \Gamma_i. \end{cases} \quad (P_N^0)\begin{cases} div\,\sigma(u_N^0) = 0 & in \quad \Omega, \\ (\sigma(u_N).n).n = \Phi.n & on \quad \Gamma_c, \\ u_N.\tau = U.\tau & on \quad \Gamma_c, \\ u_N^0 = 0 & on \quad \Gamma_i. \end{cases} \tag{6}$$

The solution of the problem (4) is recovered if:

$$\sigma(u_D).n = \sigma(u_N).n \quad on\, \Gamma_i \tag{7}$$

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With this partition, condition 7 leads to the boundary equation

$$\sigma(u_D^*).n - \sigma(u_N^*).n = \sigma(u_N^0).n - \sigma(u_D^0).n \quad on \quad \Gamma_i \tag{8}$$

Author introduce the Steklov Poincaré operator

$$S\eta = \sigma(u_D^*).n - \sigma(u_N^*).n \quad on \quad \Gamma_i$$

Author define $S_D\eta = \sigma(u_D^*).n$ and $S_N\eta = \sigma(u_N^*).n$.

Author can write the equation (8) according to the Steklov Poincaré operator:

$$S\eta = \xi \quad on \quad \Gamma_i$$

where $\xi = -(\sigma(u_D^0).n - \sigma(u_N^0).n)$

This operator, borrowed from the domain decomposition community, is widely used in ref. [13].
There are several ways to solve this linear system of equations. Here author use an iterative preconditioned gradient algorithm, which appears to be very efficient. Each iteration of the algorithm is written

$$\eta = \eta + \rho(S_D)^{-1}(S\eta - \xi),$$

where ρ is a relaxation coefficient and S_D is the preconditioning operator.

Thus each iteration requires to compute $S\eta$ by solving the two problems ?? and to solve the system $S_D\chi = S\eta$. This is achieved by solving the following problem:

$$\begin{cases} div\,\sigma(w) = 0 & in \quad \Omega, \\ \sigma(w).n = S\eta & on \quad \Gamma_i, \\ w = 0 & on \quad \Gamma_c. \end{cases} \tag{9}$$

where $\chi = w$ on Γ_i.

Now, author propose an algorithm to approximately solve the sub- Cauchy problem:

Algorithm

1. Choose arbitrary η
2. Solve problems (P_D) and (P_N).
3. solve problem (9).
4. Let $\eta = \eta + \rho\, w$
5. Go back to the first step until the stopping criteria $\|u_D - u_N\| \le \varepsilon$ is reached. (ε is a given tolerance level)

III. NUMERICAL RESULTS AND DISCUSSION

The purpose of this section is to present the numerical implementation of the boundary data recovery process described above.

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The numerical implementation is run under FreeFem software [14] based on Finite Element Method. All through this section, author consider an isotropic linear elastic material (Steel XC10 to 20° temperature) characterised by the poisson coefficient v= 0.29 and Young's modulus E = 216 GPa.

Author are concerned by a two dimentional framework corresponding to a **square hole domain**.

The partially overspecified boundary data is a synthetic one, obtained through the resolution of the following forward problem:

$$\begin{cases} div\sigma(u_0) & = 0 & in & \Omega, \\ \sigma(u_0).n & = \sigma(T)n & on & \Gamma_c, \\ u_0 & = T & on & \Gamma_i. \end{cases}$$

where $T = (Re(\frac{1}{z-a}), Im(\frac{1}{z-a}))$, z = x + iy , a =1.8 , $\partial\Omega = \Gamma_c \cup \Gamma_i$ and Γ_i being the inner circle.

Notice that we are dealing with a "rough" case, insofar as, the inffered data, are induced by a "near singular" data. The trials used in the litterature come usually from analytical reference solutions.

A. Preliminary Numerical Test

Our trial concerns the resolution of the sub-Cauchy problem in the following context: We consider a square hole domain: rectangle size: (10 * 20) with inner circle of radius R=2. The internal circle plays the role of the boundary Γ_i and the Cauchy data are donated in the **external boundary** Γ_c.

Author choose $\varepsilon = 10^{-2}$ in the stopping criteria computation are carried out with "un-noisy" data. **Figures 1-3** show the reconstructed displacement and traction on the inner boundary, whereas **Figure 4** illustrate the reconstruction of the shear stress in Γ_c. Note that the reconstruction is quite nice in Γ_i and in good agreement with exact for the shear stress.

B. Sensitivity to the Thickness

The following numerical trials are devoted to the influence of the radius of the hole on the reconstructed data. The results are summerrized. As expected the computed sub-Cauchy problem solution is better when the distance between Γ_c and Γ_i is lower. To confirm the results, **Figure 5** where author present the result for the first component of displacement on Γ_i

The same remark is true when we zoom on the shear stress.

C. Extended Domain

The following numerical experiments are inspired by Hecht [14]. In ref. [14], the authors resort to un extended domain method to illustrate its regularisation effect on their numerical data completion procedure. Their study was conducted in the framework of Laplace equation. To our opinion, the proposed method may be used in many practical situations, one can think to the data completion on rough boundary in this situation it is worthfull to extend the domain to a smooth one and to deduce the boundary conditions on the rough boundary.This trick will avoid the meshing difficulties for instance.

Another possible application may concern a void detection: If one has an apriori knowledge on the void location, the computation may be done on an extended domain, the void being detected by level lines of the displacement field [15] (for the Laplace equation). Our concern here is to illustrate the deblurring effect of this domain extension procedure.

Of course, it currently happens in practice that the data (on Γ_c) suffer from erroneous measurements, the following numerical experiments illustrates the deblurring effect of the extended domain method.

We consider a random noise of 4% added to the exact data as follows:

$$U = U + \alpha r \qquad \Phi.n = \Phi.n + \beta r$$

where (α , β) denotes the noise level relative to ($\| U \|_{L^2(\Gamma_c)}$, $\| \phi.n \|_{L^2(\Gamma_c)}$), and *r* is a random function generated by Freefem.

The boundary Γ_i is very close to the complete boundary and is then exposed to the noise contamination coming from Γ_c .The possibility of extending domain by a fictitious incomplete bounday can correct this contamination.

The exact domain is defined square by rectangle size (10 * 20) with hole of radius *R*=6 while the extended field is defined by the same rectangle, but with a hole of radius *R*=4.

Figures 6 and **7** show the reconstructed displacement for the exact and extended domain. Note that the solution computed in the real domain suffers from hard oscillations. Those obtained in the extended domain seem satisfactory.

IV. CONCLUSION

In this work the reconstruction of lacking boundary data on a part of the boundary of a body from partially-overspecified boundary conditions on another part has been investigated numerically. A domain decomposition like method has been given to describe the reconstruction process. The numerical investigation has been conducted on a "rough" configuration (i.e. the data to be recovered is not extendable on a divergence free stress field outside the domain namely within the hole), it uses FEM. The numerical section highlights the accuracy of the inverse procedure, as well as the **robustness** of the inversion process to noisy data as well as its ability to deblur noise.

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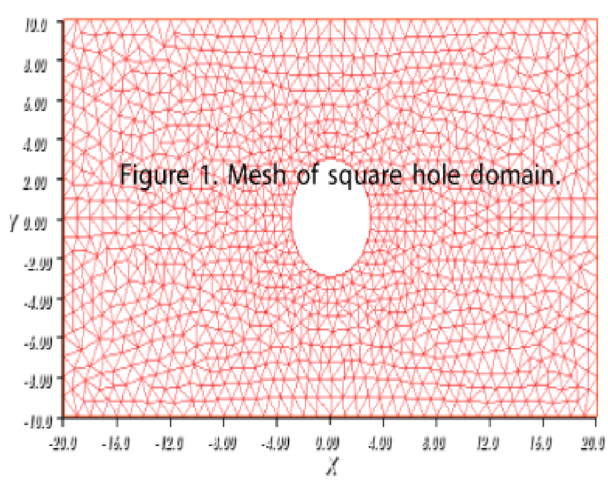
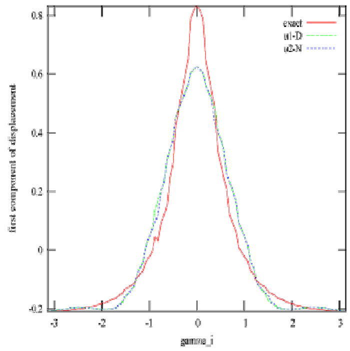
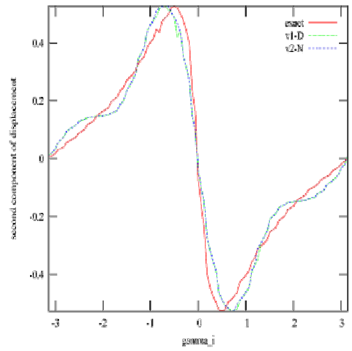
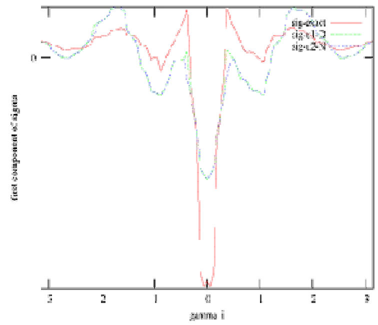
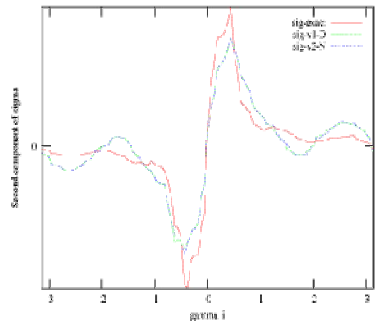
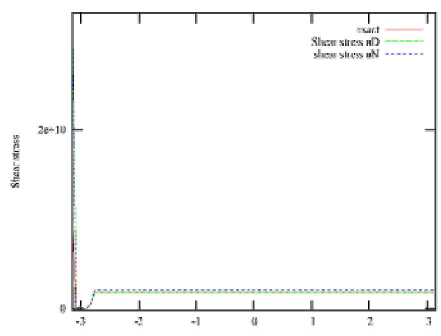
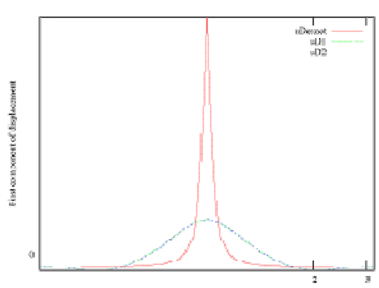
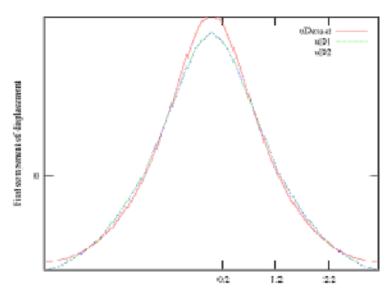


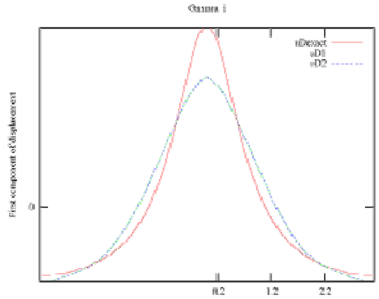
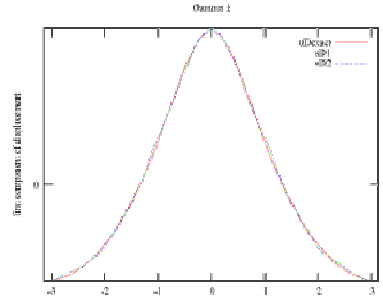
Figure 1. Mesh of square hole domain.



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