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To Study Separation Axioms in Topological Spaces

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Abstract: The purpose of this paper is to introduce weak separation axioms via sgp-closed sets in topological spaces and study some of their properties.
Keywords: T0- space or Kolmogorov Spaces, T1-Space or Frechet's Sepration axioms,T2-Space or Hausdorff Space,T3- Space Regular spaces,T4-spaces or normal spaces.

I. INTRODUCTION

In topology and related fields of mathematics, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. Some of these restrictions are given by the separation axioms.

The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The modern approach is to fix once and for all the axiomatization of topological space and then speak of kinds of topological spaces. However, the term "separation axiom" has stuck. The separation axioms are denoted with the letter "T" after the German Trennungsaxiom, which means "separation axiom."

A. T0- Spaces or Knowledge Spaces

1) *Definition:* A topological space (X,τ) is said to be T0- space if and only if given any pair of distinct points x, y of X, there exits a neightbourhood of one of them not containing the other.
2) *Example 1:* Show that every discrete space is a T0- space.
3) *Solution:* Let (X, D) be a discrete topological space and let x, y be distinct points of X. Since the space is discrete, {x} is an open nhd of x which does not countain y. If follows that (X, D) is a T0 - space.
4) *Example 2:* If X = {a,b} and τ = {Φ, {a}, X} . Is the space (X, τ) is T0 - space?
5) *Solution:* (X,τ) is T0- space as for given a,b ∈ X ⇒∃ {a} ∈ τ such that a ∈ {a}, b ∉ {a}.
6) *Theorem:* Every subspace of a T0 - space is a T0 - space and hence the property is hereditary.
7) *Proof:* Let (X,τ) be T0 - space and let (X,τ*) be a subspace of (X,τ). Let y1 , y2 be two distinct point of Y. Since Y ⊂ X, y1, y2 are also distinct points of X. Since (X,τ) is a T0 - space, there exists a τ-open nhd G of y1 not containing y2. Then G ∩ Y is τ* - open nhd G of y1 not containing y2. If follows that (X,τ*) is a T0 - space.

B. T1 - Spaces or Frechet's Separation Axiom

Definition: A topological space (X,τ) is said to be T1 space if and only if given any pair of distinct points x and y of X there exist two open sets one containing x but not y, and the other containing y but not x, that is, there exists open sets G and H such that x ∈ G but y ∉ G and y ∈ H but x∉ H.

Example 1: Show that space (R, U) is a T1 .

Solution: Let x,y be any two distinct real numbers and let y>x. Let y - x = k. Then G =] x - k/4, x + k/4, y + k/4 [are U -open sets such that x ∈ G but y ∉ G y ∈ H but x ∉ H. Hence (R, U) is a T1 -space.

C. T2 - Spaces or Hausdorff Space

Definition : A topology space (X,τ) is said to be a T2- space (Hausdorff space or separated space) iff for every pair of distinct points x, y of X, there exits dispoint neightbourhood of x and y, that is, there exist neighbourhoods N of x and M of y such that N ∩ M = Φ. If (X, τ) is a Hausdorff space, then τ is said to be a Hausdorff topology for X.

Example 1: Show that every discrete space is T2. Also show that no indiscrete space consisting of atleast two point in Hausdorff.

Solution : Since every singleton set is open in a discrete space, it follows that every pair of distinct points of such a space has disjoint neighbourhoods, namely, the singleton sets containing these points. Hence every discrete space is Hausdorff. But no indiscrete space can be Hausdorff since the whole space is the only neighbourhood of each of its points so that no two distinct points can have disjoint neighbourhoods.

D. T3 - Spaces or Regular Spaces

1) *Definition :* A topology space (X,τ) is said to be regular if and only if for every τ-closed set F and every point p ∉ F, there exist τ-open sets G and H such that

F ⊂ G, p ∈ H and G ∩ H = Φ

A regular T1 -space is called a T3 -space.

a) *Example 1:* Let X = {a, b, c} and let τ = {Φ, {a}, {b, c}, X}. Show that (X, τ) is regular space but not a T3 - space.

b) *Solution:* The closed subset are X, {b, c}, {a}, Φ.

Consider the closed set {b, c} and the point a not belonging to it. Then {b, c} and {a} are open sets such that

{b, c} ⊂ {b, c} , a ∈ {a} and {b, c} ∩ {a} = Φ

Similarly consier the closed set {a} and the point b not belonging to it. Then {a} and {b, c} are open sets such that

{a} ⊂ {a}, b ∈ {b, c} and {a} ∩ {b, c} = Φ

Again for the closed set {a} and the point c, there exist open sets {a} and {b, c} such that

{a} ⊂ {a}, c ∈ {b, c} and {a} ∩ {b, c} = Φ

It followd that (X, τ) is regular space.

Since there does not exist a τ- open set containing the point b and not containing the point c, the space is not T1 and consequently it is neither T2 nor T3.

c) *Theorem:* A topological space X is regular iff for every point x ∈ X and every neighbourhood N of x, there exists a neighbourhood M of x such that M ⊂ N. In other words, a topological space is regular iff the collection of all closed neighbourhoods oc x forms a local base at x.

d) *Proof:* If is enough to prove the theorem for open neighbourhoods.

The only 'if part': Let N be any nhd of x. Then there exists open set G such that x ∈ G ⊂ N. Since G' is closed and x ∉ G', by definition there exist open sets L and M such that G' ⊂ L, x ∈ M and L ∩ M = Φ so that M ⊂ L'.

It follows that M ⊂ (L') = L' ....(1)

Also G' ⊂ L ⇒ L' ⊂ G ⊂ N ....(2)

From (1) and (2), we get. M ⊂ N.

The only if part: Let the condition hold. Let F any closed set and let x ∉ F. Then x ∉ F'.

Since F' is open set containing x, be hypothesis there exists an open set M such that x ∈ M and M ⊂ F' ⇒F ⊂ (M'). Then (M') is an open set containing F. Also

M ∩ M' = Φ ⇒ M ∩ (M)' = Φ

Hence the space is regular.

Another Statement: A topological space (X, τ) is regular iff each open nhd of an element x ∈ X contains the closure of another open nhd of x.

T4 - sapces or Normal Space

2) *Definition :* A topological space (X, τ) is said to be normal if and only if for every pair L, M of disjoint τ -closed subsets of X there exists τ -open sets G and H such that

L ⊂ G, M ⊂ H and G ∩ H = Φ.

3) *Definition:* A normal T1 - space, is said to be a T4 -sapce.

Example : Let X = {a, b, c} and let τ = { Φ, {a}, {b, c}, X}. Show that (i) (X, τ) is normal (ii) not T2 (iii) non-Hausdorff.

Solution: Hence the only pair of disjoint closed subsets of X is {a}, {b, c},. Also {a}, {b, c} are τ-open sets such that

{a} ⊂ {a}, {b, c} ⊂ {b, c} and {a} ∩ {b, c} = Φ

It follows that the space is normal. Again there does not exists a τ-open set containing the point b and not containing point c.

Hence the space is not T1 and consequently it is neither T2 nor T3. Thus the space is non- T3 and non - Hausdorff space.

II. CONCLUSION

A property P of a topological space X is said to be weakly hereditary ig and only if every closed subspace of X has the property P. Thus Lindelofbess, normality and compactness are weakly hereditary properties.

III. RESULT AND DISCUSSION

Theorem 1 : Every T4 -space is also a T3 -space.

Proof : Let (X, τ) be a T4 -space. Then (X, τ) is normal as well as T1. To show that the space is T3, it suffices to show that the space is regular. Let F be a τ -closed subset of X and let x be a point of X such that x ∉ F. Since (X, τ) is a T1 -space, {x} is a closed subset of X such that {x} ∩ F = Φ. Then by normality, there open sets G and H such that {x} ⊂ G, F ⊂ H and G ∩ H = Φ. Alsi {x} ⊂ G ⇒x ∉ G.

Thus there exist open sets G, H such that x ∈ G, F ⊂ H and G ∩ H = Φ. It follows that the space (X, τ) is regular.

Theorem 2 : Every compact Hausdorff space is normal space (T4).

Proof : Let (X, τ) be a compact Hausdorff space and let L, M be a pair of disjoint closed subsets of X. The space (X, τ) is regular and so for each x

∈ L, there exists τ-open sets Gx and Hx such that x ∈ Gx, M ⊂ Hx and Gx ⊂ Hx = Φ. Then the collection {Gx : x ∈ L} is an open cover of L.

L is compact subspace of X. Hence there exists a finite number points, xi, i = 1,2,.....n in L such that

L ⊂ ∪ {Gxi : i = 1,2,....n}

Let G = ∪ {Gxi : i = 1,2,....n} and H = ∩ {Hxi : i = 1,2,....n}

Then G, H are τ -open sets such that

L ⊂ G, M ⊂ H and G ∩ H = Φ

Hence (X, τ) is normal. Since every Hausdorff space is a T- space. It follows that (X, τ) is also a T1 -space.

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