iJRASET
International Journal For Research in Applied Science and Engineering Technology



INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 7 **Issue:** VII **Month of publication:** July 2019

DOI: http://doi.org/10.22214/ijraset.2019.7104

www.ijraset.com

Call: 08813907089 | E-mail ID: ijraset@gmail.com

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.177
Volume 7 Issue VII, July 2019- Available at www.ijraset.com

On the Positive Pell Equation

$y^2 = 8x^2 + 49$

A. Kavitha[1], S. Karpagam[2]
[1]*Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil nadu, India.*
[2]*Mphil Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil nadu, India.*

Abstract: ***The binary quadratic Diophantine equation represented by the positive pellian $y^2 = 8x^2 + 49$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.***
Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4].For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 8x^2 + 49$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 8x^2 + 49 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 2\,,\ y_0 = 9\ \ D = 8$$

The pellian equation is

$$y^2 = 8x^2 + 1 \tag{2}$$

The initial solution of pellian equation is

$$\tilde{x}_0 = 1, \tilde{y}_0 = 3,$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{8}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (3+\sqrt{8})^{n+1} + (3-\sqrt{8})^{n+1}$$

$$g_n = (3+\sqrt{8})^{n+1} - (3-\sqrt{8})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = f_n + \frac{9}{2\sqrt{8}} g_n$$

$$y_{n+1} = \frac{9}{2} f_n + \frac{8}{\sqrt{8}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

©IJRASET: All Rights are Reserved

ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.177
Volume 7 Issue VII, July 2019- Available at www.ijraset.com

$$x_{n+3} - 6x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 6y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	2	9
1	15	43
2	88	249
3	513	1451
4	2990	8457

From the above table, we observe some interesting relations among the solutions which are presented below.

A. Both x_n *and* y_n values are odd and even.

B. Each of the following expression is a nasty number:

1) $\frac{6}{49}\left[98+18y_{2n+2}-32x_{2n+2}\right]$

2) $\frac{6}{49}\left[98+18x_{2n+3}-86x_{2n+2}\right]$

3) $\frac{6}{147}\left[294+9x_{2n+4}-249x_{2n+2}\right]$

4) $\frac{6}{147}\left[294+18y_{2n+3}-240x_{2n+2}\right]$

5) $\frac{6}{833}\left[1666+18y_{2n+4}-1408x_{2n+2}\right]$

6) $\frac{6}{147}\left[294+86y_{2n+2}-32x_{2n+3}\right]$

7) $\frac{6}{833}\left[1666+498y_{2n+2}-32x_{2n+4}\right]$

8) $\frac{6}{392}\left[784+240y_{2n+2}-32y_{2n+3}\right]$

9) $\frac{6}{2352}\left[4704+1408y_{2n+2}-32y_{2n+4}\right]$

10) $\frac{6}{49}\left[98+86y_{2n+3}-240x_{2n+3}\right]$

11) $\frac{6}{49}\left[98+86x_{2n+4}-498x_{2n+3}\right]$

12) $\frac{6}{147}\left[294+86y_{2n+4}-1408x_{2n+3}\right]$

©IJRASET: All Rights are Reserved

13) $\frac{6}{147}[294+498y_{2n+3}-240x_{2n+4}]$

14) $\frac{6}{49}[98+498y_{2n+4}-1408x_{2n+4}]$

15) $\frac{6}{392}[784+1408y_{2n+3}-240y_{2n+4}]$

C. Each of the following expressions is a cubical integer.

1) $\frac{1}{49}[18y_{3n+3}-32x_{3n+3}+54y_{n+1}-96x_{n+1}]$

2) $\frac{1}{49}[18x_{3n+4}-86x_{3n+3}+54x_{n+2}-258x_{n+1}]$

3) $\frac{1}{147}[9x_{3n+5}-249x_{3n+3}+27x_{n+3}-747x_{n+1}]$

4) $\frac{1}{147}[18y_{3n+4}-240x_{3n+3}+54y_{n+2}-720x_{n+1}]$

5) $\frac{1}{833}[18y_{3n+5}-1408x_{3n+3}+54y_{n+3}-4224x_{n+1}]$

6) $\frac{1}{147}[86y_{3n+3}-32x_{3n+4}+258y_{n+1}-96x_{n+2}]$

7) $\frac{1}{833}[498y_{3n+3}-32x_{3n+5}+1494y_{n+1}-96x_{n+3}]$

8) $\frac{1}{392}[240y_{3n+3}-32y_{3n+4}+720y_{n+1}-96y_{n+2}]$

9) $\frac{1}{2352}[1408y_{3n+3}-32y_{3n+5}+4224y_{n+1}-96y_{n+3}]$

10) $\frac{1}{49}[86x_{3n+5}-498x_{3n+4}+258x_{n+3}-1494x_{n+2}]$

11) $\frac{1}{49}[86y_{3n+4}-240x_{3n+4}+258y_{n+2}-720x_{n+2}]$

12) $\frac{1}{147}[86y_{3n+5}-1408x_{3n+4}+258y_{n+3}-4224x_{n+2}]$

13) $\frac{1}{147}[498y_{3n+4\backslash}-240x_{3n+5}+1494y_{n+2}-720x_{n+3}]$

14) $\frac{1}{49}[498y_{3n+5\backslash}-1408x_{3n+5}+1494y_{n+3}-4224x_{n+3}]$

15) $\frac{1}{392}[1408y_{3n+4\backslash}-240y_{3n+5}+4224y_{n+2}-720y_{n+3}]$

©IJRASET: All Rights are Reserved

D. Each of the following expressions is a biquadratic integer.

16) $\frac{1}{49}\left[18y_{4n+4}-32x_{4n+4}+72y_{2n+2}-128x_{2n+2}+294\right]$

17) $\frac{1}{49}\left[18x_{4n+5}-86x_{4n+4}+72x_{2n+3}-344x_{2n+2}+294\right]$

18) $\frac{1}{147}\left[9x_{4n+6}-249x_{4n+4}+36x_{2n+4}-996x_{2n+2}+882\right]$

19) $\frac{1}{147}\left[18y_{4n+5}-240x_{4n+4}+72y_{2n+3}-960x_{2n+2}+882\right]$

20) $\frac{1}{833}\left[18y_{4n+6}-1408x_{4n+4}+72y_{2n+4}-5632x_{2n+2}+4998\right]$

21) $\frac{1}{147}\left[86y_{4n+4}-32x_{4n+5}+344y_{2n+2}-128x_{2n+3}+882\right]$

22) $\frac{1}{833}\left[498y_{4n+4}-32x_{4n+6}+1992y_{2n+2}-128x_{2n+4}+4998\right]$

23) $\frac{1}{392}\left[240y_{4n+4}-32y_{4n+5}+960y_{2n+2}-128y_{2n+3}+2352\right]$

24) $\frac{1}{2352}\left[1408y_{4n+4}-32y_{4n+6}+5632y_{2n+2}-128y_{2n+4}+14112\right]$

25) $\frac{1}{49}\left[86x_{4n+6}-498x_{4n+5}+344x_{2n+4}-1992x_{2n+3}+294\right]$

26) $\frac{1}{49}\left[86y_{4n+5}-240x_{4n+5}+344y_{2n+3}-960x_{2n+3}+294\right]$

27) $\frac{1}{147}\left[86y_{4n+6}-1408x_{4n+5}+344y_{2n+4}-5632x_{2n+3}+882\right]$

28) $\frac{1}{147}\left[498y_{4n+5}-240x_{4n+6}+1992y_{2n+3}-960x_{2n+4}+882\right]$

29) $\frac{1}{49}\left[498y_{4n+6}-1408x_{4n+6}+1992y_{2n+4}-5632x_{2n+4}+294\right]$

30) $\frac{1}{392}\left[1408y_{4n+5}-240y_{4n+6}+5632y_{2n+3}-960y_{2n+4}+2352\right]$

E. Each of the following expression is a quintic integer:

1) $\frac{1}{49}\left[18y_{5n+5}-32x_{5n+5}+90y_{3n+3}-160x_{3n+3}+180y_{n+1}-320x_{n+1}\right]$

2) $\frac{1}{49}\left[18x_{5n+6}-86x_{5n+5}+90x_{3n+4}-430x_{3n+3}+180x_{n+2}-860x_{n+1}\right]$

3) $\frac{1}{147}\left[9x_{5n+7}-249x_{5n+5}-45x_{3n+5}-1245x_{3n+3}+90x_{n+3}-2490x_{n+1}\right]$

©IJRASET: All Rights are Reserved

4) $\frac{1}{147}[18y_{5n+6} - 240x_{5n+5} + 90y_{3n+4} - 1200x_{3n+3} + 180y_{n+2} - 2400x_{n+1}]$

5) $\frac{1}{833}[18y_{5n+7} - 1408x_{5n+5} + 90y_{3n+5} - 7040x_{3n+3} + 180y_{n+3}]$

6) $\frac{1}{147}[86y_{5n+5} - 32x_{5n+6} + 430y_{3n+3} - 160x_{3n+4} + 860y_{n+1} - 320x_{n+2}]$

7) $\frac{1}{833}[498y_{5n+5} - 32x_{5n+7} + 2490y_{3n+3} - 160x_{3n+5} + 4980y_{n+1} - 320x_{n+3}]$

8) $\frac{1}{392}[240y_{5n+5} - 32y_{5n+6} + 1200y_{3n+3} - 160y_{3n+4} + 2400y_{n+1} - 320y_{n+2}]$

9) $\frac{1}{2352}[1408y_{5n+5} - 32y_{5n+7} + 7040y_{3n+3} - 160y_{3n+5} + 14080y_{n+1} - 320y_{n+3}]$

10) $\frac{1}{49}[86x_{5n+7} - 498x_{5n+6} + 430x_{3n+5} - 2490x_{3n+4} + 860x_{n+3} - 4980x_{n+2}]$

11) $\frac{1}{49}[86y_{5n+6} - 240x_{5n+6} + 430y_{3n+4} - 1200x_{3n+4} + 860y_{n+2} - 2400x_{n+2}]$

12) $\frac{1}{147}[86y_{5n+7} - 1408x_{5n+6} + 430y_{3n+5} - 7040x_{3n+4} + 860y_{n+3} - 14080x_{n+2}]$

13) $\frac{1}{147}[498y_{5n+6} - 240x_{5n+7} + 2490y_{3n+4} - 1200x_{3n+5} + 4980y_{n+2} - 2400x_{n+3}]$

14) $\frac{1}{49}[498y_{5n+7} - 1408x_{5n+7} + 2490y_{3n+5} - 7040x_{3n+5} + 4980y_{n+3} - 14080x_{n+3}]$

15) $\frac{1}{392}[1408y_{5n+6} - 240y_{5n+7} + 7040y_{3n+4} - 1200y_{3n+5} + 14080y_{n+2} - 24840y_{n+3}]$

F. Relations among the solutions are given below:

1) $x_{n+2} = y_{n+1} + 3x_{n+1}$

2) $x_{n+3} = 6y_{n+1} + 17x_{n+1}$

3) $y_{n+2} = 3y_{n+1} + 8x_{n+1}$

4) $y_{n+3} = 17y_{n+1} + 48x_{n+1}$

5) $x_{n+3} = 6x_{n+2} - x_{n+1}$

6) $y_{n+2} = 3x_{n+2} - x_{n+1}$

7) $y_{n+3} = 17x_{n+2} - 3x_{n+1}$

8) $2y_{n+2} = x_{n+3} - x_{n+1}$

9) $6y_{n+3} = 17x_{n+3} - x_{n+1}$

10) $3y_{n+3} = 17y_{n+2} + 8x_{n+1}$

11) $3x_{n+3} = y_{n+1} + 17x_{n+2}$

12) $3y_{n+2} = y_{n+1} + 8x_{n+2}$

©IJRASET: All Rights are Reserved

13) $y_{n+3} = y_{n+1} + 16x_{n+2}$

14) $17y_{n+2} = 3y_{n+1} + 8x_{n+3}$

15) $17y_{n+3} = y_{n+1} + 48x_{n+3}$

16) $y_{n+3} = 6y_{n+2} - y_{n+1}$

17) $y_{n+2} = x_{n+3} - 3x_{n+2}$

18) $y_{n+3} = 3x_{n+3} - x_{n+2}$

19) $y_{n+3} = 3y_{n+2} + 8x_{n+2}$

20) $3y_{n+3} = y_{n+2} + 8x_{n+3}$

III.REMARKABLE OBSERVATION

A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 8X^2 = 9604$	$(18x_{n+1} - 4y_{n+1}, 18y_{n+1} - 32x_{n+1})$
2	$Y^2 - 8X^2 = 9604$	$(30x_{n+1} - 4x_{n+2}, 18x_{n+2} - 86x_{n+1})$
3	$Y^2 - 8X^2 = 86436$	$(88x_{n+1} - 2x_{n+3}, 9x_{n+3} - 249x_{n+1})$
4	$Y^2 - 8X^2 = 86436$	$(86x_{n+1} - 4y_{n+2}, 18y_{n+2} - 240x_{n+1})$
5	$Y^2 - 8X^2 = 2775556$	$(498x_{n+1} - 4y_{n+3}, 18y_{n+3} - 1408x_{n+1})$
6	$Y^2 - 8X^2 = 86436$	$(18x_{n+2} - 30y_{n+1}, 86y_{n+1} - 32x_{n+2})$
7	$Y^2 - 8X^2 = 2775556$	$(18x_{n+3} - 176y_{n+1}, 498y_{n+1} - 32x_{n+3})$
8	$Y^2 - 8X^2 = 614656$	$(18y_{n+2} - 86y_{n+1}, 240y_{n+1} - 32y_{n+2})$
9	$Y^2 - 8X^2 = 22127616$	$(18y_{n+3} - 498y_{n+1}, 1408y_{n+1} - 32y_{n+3})$
10	$Y^2 - 8X^2 = 9604$	$(176x_{n+2} - 30x_{n+3}, 86x_{n+3} - 498x_{n+2})$
11	$Y^2 - 8X^2 = 9604$	$(86x_{n+2} - 30y_{n+2}, 86y_{n+2} - 240x_{n+2})$
12	$Y^2 - 8X^2 = 86436$	$(498x_{n+2} - 30y_{n+3}, 86y_{n+3} - 1408x_{n+2})$
13	$Y^2 - 8X^2 = 86436$	$(86x_{n+3} - 176y_{n+2}, 498y_{n+2} - 240x_{n+3})$
14	$Y^2 - 8X^2 = 9604$	$(498x_{n+3} - 176y_{n+3}, 498y_{n+3} - 1408x_{n+3})$
15	$Y^2 - 8X^2 = 614656$	$(86y_{n+3} - 498y_{n+2}, 1408y_{n+2} - 240y_{n+3})$

©IJRASET: All Rights are Reserved

B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$49Y-8X^2=9604$	$(18x_{n+1}-4y_{n+1},18y_{2n+2}-32x_{2n+2}+98)$
2	$49Y-8X^2=9604$	$(30x_{n+1}-4x_{n+2},18x_{2n+3}-86x_{2n+2}+98)$
3	$147Y-8X^2=86436$	$(88x_{n+1}-2x_{n+3},9x_{2n+4}-249x_{2n+2}+294)$
4	$147Y-8X^2=86436$	$(86x_{n+1}-4y_{n+2},18y_{2n+3}-240x_{2n+2}+294)$
5	$833Y-8X^2=2775556$	$(498x_{n+1}-4y_{n+3},18y_{2n+4}-1408x_{2n+2}+1666)$
6	$147Y-8X^2=86436$	$(18x_{n+2}-30y_{n+1},86y_{2n+2}-32x_{2n+3}+294)$
7	$833Y-8X^2=2775556$	$(18x_{n+3}-176y_{n+1},498y_{2n+2}-32x_{2n+4}+1666)$
8	$392Y-8X^2=614656$	$(18y_{n+2}-86y_{n+1},240y_{2n+2}-32y_{2n+3}+784)$
9	$2352Y-8X^2=22127616$	$(18y_{n+3}-498y_{n+1},1408y_{2n+2}-32y_{2n+4}+4704)$
10	$49Y-8X^2=9604$	$(176x_{n+2}-30x_{n+3},86x_{2n+4}-498x_{2n+3}+98)$
11	$49Y-8X^2=9604$	$(86x_{n+2}-30y_{n+2},86y_{2n+3}-240x_{2n+3}+98)$
12	$147Y-8X^2=86436$	$(498x_{n+2}-30y_{n+3},86y_{2n+4}-1408x_{2n+3}+294)$
13	$147Y-8X^2=86436$	$(86x_{n+3}-176y_{n+2},498y_{2n+3}-240x_{2n+4}+294)$
14	$49Y-8X^2=9604$	$(498x_{n+3}-176y_{n+3},498y_{2n+4}-1408x_{2n+4}+98)$
15	$392Y-8X^2=614656$	$(86y_{n+3}-498y_{n+2},1408y_{2n+3}-240y_{2n+4}+784)$

1) Special Cases

a) $P_y^{10}\left(t_{3,x+1}\right)^2=153P_x^6\left(t_{3,y}\right)^2+8\left(t_{3,y}\right)^2\left(t_{3,x+1}\right)^2$

b) $9P_y^6\left(t_{3,x}\right)^2=17P_x^{10}\left(t_{3,y+1}\right)^2+8\left(t_{3,x}\right)^2\left(t_{3,y+1}\right)^2$

c) $P_y^{10}\left(t_{3,2x-2}\right)^2=17\left(6P_{x-1}^4\right)^2\left(t_{3,y}\right)^2+8\left(t_{3,y}\right)^2\left(t_{3,2x-2}\right)^2$

d) $36P_{y-1}^8\left(t_{3,x}\right)^2=17P_x^{10}\left(t_{3,2y-2}\right)^2+8\left(t_{3,x}\right)^2\left(t_{3,2y-2}\right)^2$

e) $9P_y^6\left(t_{3,2x-2}\right)^2=17\left(36P_{x-1}^8\right)\left(t_{3,y+1}\right)^2+8\left(t_{3,2x-2}\right)^2\left(t_{3,y+1}\right)^2$

f) $\left(6P_{y-1}^4\right)^2\left(t_{3,x+1}\right)^2=17\left(3P_x^3\right)^2\left(t_{3,2y-2}\right)^2+8\left(t_{3,x+1}\right)^2\left(t_{3,2y-2}\right)^2$

©IJRASET: All Rights are Reserved

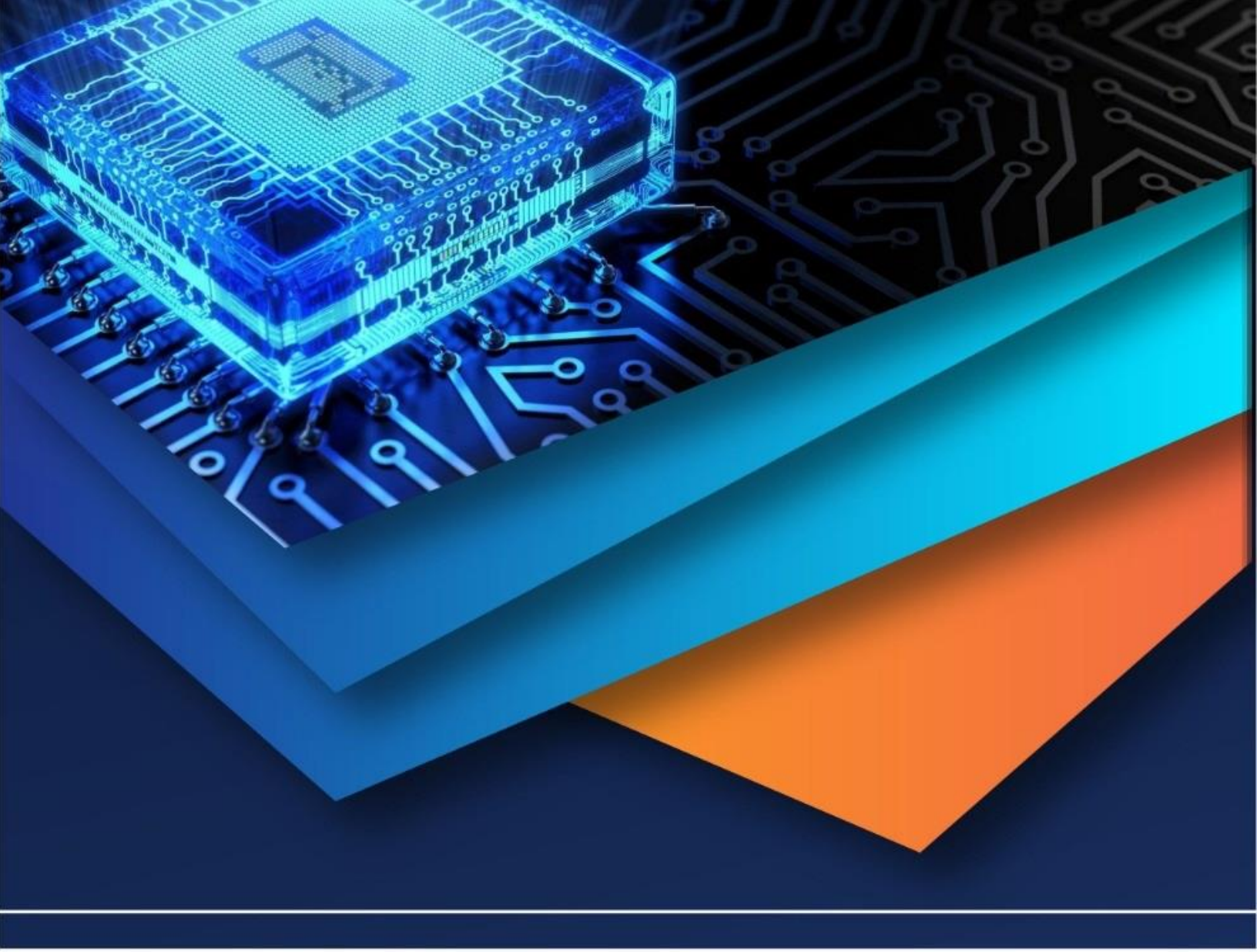
IV. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2 = 8x^2 + 49$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties

REFERENCES

[1] L.E.Dickson, History of theory of number, Chelsa publishing company, vol-2, (1952) New York.
[2] L.J.Mordel, Diophantine equations, Academic Press, (1969) New York.
[3] S.J.Telang, Number Theory, Tata McGraw Hill Publishing company Limited,(2000) New Delhi.
[4] D.M.Burton, Elementary Number Theory, Tata McGraw Hill Publishing company Limited,(2002) New Delhi.
[5] M.A.Gopolan, s.vidhyalakshmi and A.Kavitha, on the integral solutions of the Binary quadratic Equation $X^2 = 4(K^2+1)Y^2 + 4^t$, Bulletin of Mathematics and Statistics Research, 2(1)(2014)42-46.
[6] S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, On the binary quadratic Diophantine equation $x^2 - 3xy + y^2 + 33x = 0$, International Journal of Scientific Engineering and Applied Science, 1(4) (2015) 222-225.
[7] M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, observations on the Hyperbola $10y^2 - 3x^2 = 13$, Archimedes J.Math.,3(1) (2013) 31-34.
[8] S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Observations on the Hyperbola $ax^2 - (a+1)y^2 = 3a - 1$, Discovery, 4(10) (2013) 22-24.
[9] S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Integral points on the Hyperbola $x^2 - 4xy + y^2 + 15x = 0$, Diophantus J.Maths,1(7) (2014) 338-340.
[10] M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, on the integral solutions Of the binary quadratic equation $x^2 = 15y^2 - 11^t$, scholars journal of Engineering and technology, 2(2A) (2014) 156-158.

©IJRASET: All Rights are Reserved



10.22214/IJRASET

45.98

IMPACT FACTOR: 7.129

IMPACT FACTOR: 7.429

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Call : 08813907089 (24*7 Support on Whatsapp)