



IJRASET

International Journal For Research in
Applied Science and Engineering Technology



INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 3

Issue: V

Month of publication: May 2015

DOI:

www.ijraset.com

Call:  08813907089

E-mail ID: ijraset@gmail.com

The bounds of crossing number in complete bipartite graphs

M. Malathi¹, Dr. J. Ravi Sankar², Dr. N. Selvi³

¹Department of Mathematics, Saradha Gangadharan College, Pondicherry, India – 605 004.

²School of Advanced Sciences, VIT University, Vellore, India – 632 014.

³Department of Mathematics, ADM College for Women, Nagapattinam, India-611001

Abstract—We compare the lower bound of crossing number of bipartite and complete bipartite graph with Zarankiewicz conjecture and we illustrate the possible upper bound by a modified Zarankiewicz conjecture.

Keywords—complete bipartite graphs, crossing numbers

I. INTRODUCTION

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$.

The crossing number of a graph G , denoted by $Cr(G)$, is the minimum number of crossings in a drawing of G in the plane [2,3,4].

The crossing number of the complete bipartite graph [7] was first introduced by Paul Turan, by his brick factory problem.

In 1954, Zarankiewicz conjectured [8] that,

$$Z(m, n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

Where m and n are vertices.

Later, Richard Guy shown that the conjecture doesnot holds for all m, n . Then in 1970 D.J.Kleitman proved that Zarankiewicz conjecture holds for $\text{Min}(m, n) \leq 6$.

A good drawing of a graph G is a drawing where the edges are non-self-intersecting in which any two edges have atmost one point in common other than end vertex. That is, a crossing is a point of intersection of two edges and no three edges intersect at a common point. So a good drawing is a crossing free drawing by arriving at a planar graph.

The crossing number is an important measure of the non-planarity of a graph. Therefore this application can be widely applied in all real time problems.

II. A MODIFIED ZARANKIEWICZ CONJECTURE

For any complete bipartite graphs with 'n' vertices,

$$Z(n, n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^2 \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right]$$

By using this conjecture, we can get all the possible number of crossings between every vertices for a given 'n', without any good drawing D . This reverse way of finding the crossings facilitates for large 'n' by without drawing the graph, we can get all possible crossings between every edges.

The best known lower bound on general case for all $m, n \in \mathbb{N}$ which was proved by D.J.Kleitman [1] in the following theorem. That is,

Theorem1[6]:

$$cr(K_{5, n}) \geq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$
$$cr(K_{6, n}) \geq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

From this he deduced that

$$cr(K_{m,n}) \geq \frac{1}{5}m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

Theorem2:

$$\text{For } m > n, cr(Z(m,n)) \geq cr(K_{m,n}) \leq cr(K_{n,m}).$$

Proof:

From theorem 1,

$$cr(K_{m,n}) \geq \frac{1}{5}m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

By definition,

$$Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

We can prove the theorem by induction. Since in $cr(K_{m,n})$, there are ${}^m C_5 K_{5,n}$ subgraphs of $K_{m,n}$ with the partite with 'n' vertices in $K_{5,n}$. So we shall obtain the lower bound of $cr(K_{m,n})$ for $m \geq 5$ and $n \geq 3$.

Case(i): Let $n=3$.

Subcase(i): $m=5$,

$$cr(Z(5,3)) = 2.2.1.1 = 4$$

$$cr(K_{5,3}) = \frac{1}{5}.5.4.1.1 = 4$$

$$cr(K_{3,5}) = \frac{1}{5}.3.2.2.2 = \frac{24}{5} = 4.8$$

$$\therefore cr(Z(5,3)) \geq cr(K_{5,3}) \leq cr(K_{3,5})$$

Subcase(ii): $m=6$,

$$cr(Z(6,3)) = 3.2.1.1 = 6$$

$$cr(K_{6,3}) = \frac{1}{5}.6.5.1.1 = 6$$

$$cr(K_{3,6}) = \frac{1}{5}.3.2.3.2 = \frac{36}{5} = 7.2$$

$$\therefore cr(Z(6,3)) \geq cr(K_{6,3}) \leq cr(K_{3,6})$$

Subcase(ii): $m=7$,

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

$$cr(Z(7,3)) = 3.3.1.1 = 9$$

$$cr(K_{7,3}) = \frac{1}{5} \cdot 7.6.1.1 = \frac{42}{5} = 8.4$$

$$cr(K_{3,7}) = \frac{1}{5} \cdot 3.2.3.3 = \frac{54}{5} = 10.8$$

$$\therefore cr(Z(7,3)) \geq cr(K_{7,3}) \leq cr(K_{3,7})$$

$$\Rightarrow cr(Z(m,3)) \geq cr(K_{m,3}) \leq cr(K_{3,m})$$

Case(ii): Let $n=4$.

Subcase(i): $m=5$,

$$cr(Z(5,4)) = 2.2.2.1 = 8$$

$$cr(K_{5,4}) = \frac{1}{5} \cdot 5.4.2.1 = 8$$

$$cr(K_{4,5}) = \frac{1}{5} \cdot 4.3.2.2 = \frac{48}{5} = 9.8$$

$$\therefore cr(Z(5,4)) \geq cr(K_{5,4}) \leq cr(K_{4,5})$$

Subcase(ii): $m=6$,

$$cr(Z(6,4)) = 3.2.2.1 = 12$$

$$cr(K_{6,4}) = \frac{1}{5} \cdot 6.5.2.1 = 12$$

$$cr(K_{4,6}) = \frac{1}{5} \cdot 4.3.3.2 = \frac{72}{5} = 14.4$$

$$\therefore cr(Z(6,4)) \geq cr(K_{6,4}) \leq cr(K_{4,6})$$

Subcase(ii): $m=7$,

$$cr(Z(7,4)) = 3.3.2.1 = 18$$

$$cr(K_{7,4}) = \frac{1}{5} \cdot 7.6.2.1 = \frac{84}{5} = 16.8$$

$$cr(K_{4,7}) = \frac{1}{5} \cdot 4.3.3.3 = \frac{108}{5} = 21.6$$

$$\therefore cr(Z(7,4)) \geq cr(K_{7,4}) \leq cr(K_{4,7})$$

$$\Rightarrow cr(Z(m,4)) \geq cr(K_{m,4}) \leq cr(K_{4,m})$$

In general,

$$cr(Z(m,n)) \geq cr(K_{m,n}) \leq cr(K_{n,m}).$$

We also observe that the following inequality ,

$$2cr(Z(m,n+1)) \geq 2cr(K_{m,n+1}) \leq 2cr(K_{n+1,m})$$

Also holds good for the above cases.

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

Hence the proof.

Theorem 2:

For $m = n$, $cr(Z(m, n)) \geq cr(K_{m,n})$. That is,

$$\frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^2 \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] =$$

$$\left\lfloor \frac{n}{2} \right\rfloor^4 \geq \frac{1}{5} n(n-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

Proof:

When $m = n$, $Z(n, n)$ is a complete bipartite graph.

By theorem 1,

$$cr(K_{m,n}) \geq \frac{1}{5} n(n-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

We shall prove the theorem for large sufficiently larger 'n' and hence deducing the result for subsequent small 'n'.

Case(i):

$$cr(K_{11,11}) = 625$$

$$= \frac{1}{2} \left[\begin{matrix} 5(5)^2 + 4(5)^2 + 4(5)^2 + 4(5)^2 + 4(5)^2 + 4(5)^2 \\ + 5(5)^2 + 5(5)^2 + 5(5)^2 + 5(5)^2 + 5(5)^2 \end{matrix} \right]$$

$$= \frac{1}{2} (5)^2 [6.5 + 5.4]$$

$$= \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^2 \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right]$$

$$= \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^3 \left[2 \left\lfloor \frac{n}{2} \right\rfloor \right]$$

$$= \left\lfloor \frac{n}{2} \right\rfloor^4$$

$$cr(Z(11,11)) = 550 = \frac{1}{5} \cdot 11 \cdot 10 \cdot 5 \cdot 5$$

$$= \frac{1}{5} \cdot 11(11-1) \left\lfloor \frac{11}{2} \right\rfloor \left\lfloor \frac{11-1}{2} \right\rfloor$$

$$= \frac{1}{5} \cdot n(n-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$\Rightarrow cr(Z(11,11)) \geq cr(K_{11,11})$$

Case(ii):

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

$$cr(Z(9,9)) = 256 = \frac{1}{5} \cdot 9 \cdot 8 \cdot 4 \cdot 4$$

$$= \frac{1}{5} \cdot 9(9-1) \left\lfloor \frac{9}{2} \right\rfloor \left\lfloor \frac{9-1}{2} \right\rfloor$$

$$= \frac{1}{5} \cdot n(n-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$cr(K_{9,9}) = 256$$

$$= \frac{1}{2} \left[4(4)^2 + 3(4)^2 + 3(4)^2 + 3(4)^2 + 3(4)^2 + \right. \\ \left. 4(4)^2 + 4(4)^2 + 4(4)^2 + 4(4)^2 \right]$$

$$= \frac{1}{2} (4)^2 [5 \cdot 4 + 4 \cdot 3]$$

$$= \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^2 \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \Rightarrow cr(Z(9,9)) \geq cr(K_{9,9})$$

$$= \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^3 \left[2 \left\lfloor \frac{n}{2} \right\rfloor \right]$$

$$= \left\lfloor \frac{n}{2} \right\rfloor^4$$

In general,

$$cr(Z(m,n)) \geq cr(K_{m,n})$$

Hence the proof.

III. CONCLUSION

We have given an alternate way of finding crossings in complete bipartite graphs. We also proved in bipartite graphs, the best lower bound of $cr(K_{m,n})$ will always be a lower bound until 'm' and 'n' are altered.

REFERENCES

- [1] Daniel J. Kleitman, The crossing number of $K_{5,n}$, J. Combinatorial Theory 9(1970),315-323.
- [2] D.R. Woodall, Cyclic-order graph and Zarankiewicz's crossing number conjecture, J.Graph Theory. 17 (1994), 657-671.
- [3] P. Erdos, R.P. Guy, Crossing number problem, American Mathematical Monthly 80 (1973), 52-58.
- [4] R.K. Guy, The decline and fall of Zarankiewicz's theorem, Proof techniques in Graph Theory, F. Harary, ed. Academic Press, New York(1969),63-69.
- [5] R.K. Guy and T. Jenkins, The toroidal crossing number of $K_{m;n}$, J. Combinatorial Theory.6(1969),235-250.
- [6] N.H. Nahas, On the crossing number of $K_{m;n}$. Electron. J. Combin. 10:N8,(2003).
- [7] R. Bruce Richter, Carsten Thomassen, Relations between crossing numbers of complete and complete bipartite graphs, The American Mathematical Monthly 104 (1997),131-137.
- [8] K. Zarankiewicz, On a problem of P. Turan concerning graphs, Fund. Math. 41 (1954),137-145.



10.22214/IJRASET



45.98



IMPACT FACTOR:
7.129



IMPACT FACTOR:
7.429



INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Call : 08813907089  (24*7 Support on Whatsapp)