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# The bounds of crossing number in complete bipartite graphs 

M. Malathi ${ }^{1}$, Dr. J. Ravi Sankar ${ }^{2}$, Dr. N. Selvi ${ }^{3}$<br>${ }^{l}$ Department of Mathematics. Saradha Gangadharan College, Pondicherry, India - 605004.<br>${ }^{2}$ School of Advanced Sciences, VIT University, Vellore, India - 632014.<br>${ }^{3}$ Department of Mathematics, ADM College for Women, Nagapattinam, India-611001


#### Abstract

We compare the lower bound of crossing number of bipartite and complete bipartite graph with Zarankiewicz conjecture and we illustrate the possible upper bound by a modified Zarankiewicz conjecture. Keywords-complete bipartite graphs, crossing numbers


## I. INTRODUCTION

Let $\mathrm{G}=(\mathrm{V}, \mathrm{E})$ be a simple connected graph with vertex set $\mathrm{V}(\mathrm{G})$ and edge set $\mathrm{E}(\mathrm{G})$.
The crossing number of a graph G , denoted by $\mathrm{Cr}(\mathrm{G})$, is the minimum number of crossings in a drawing of G in the plane[2,3,4].
The crossing number of the complete bipartite graph [7] was first introduced by Paul Turan, by his brick factory problem.
In 1954, Zarankiewicz conjectured [8] that,

$$
\left.\mathrm{Z}(\mathrm{~m}, \mathrm{n})=\left\lfloor\frac{m}{2}\right\rfloor\left[\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor \frac{n-1}{2}\right\rfloor
$$

Where m and n are vertices.
Later, Richard Guy shown that the conjecture doesnot holds for all m,n. Then in 1970 D.J.Kleitman proved that Zarankiewicz conjecture holds for $\operatorname{Min}(\mathrm{m}, \mathrm{n}) \leq 6$.
A good drawing of a graph G is a drawing where the edges are non-self-intersecting in which any two edges have atmost one point in common other than end vertex. That is, a crossing is a point of intersection of two edges and no three edges intersect at a common point. So a good drawing is a crossing free drawing by arriving at a planar graph.
The crossing number is an important measure of the non-planarity of a graph. Therefore this application can be widely applied in all real time problems.

## II. A MODIFIED ZARANKIEWICZ CONJECTURE

For any complete bipartite graphs with ' $n$ ' vertices,

$$
Z(n, n)=\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^{2}\left[\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)+\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right]
$$

By using this conjecture, we can get all the possible number of crossings between every vertices for a given ' n ', without any good drawing D . This reverse way of finding the crossings facilitates for large ' $n$ ' by without drawing the graph, we can get all possible crossings between every edges.

The best known lower bound on general case for all $\mathrm{m}, \mathrm{n} \in \mathrm{N}$ which was proved by D.J.Kleitman [1] in the following theorem. That is,
Theorem1[6]:

$$
\begin{aligned}
& \operatorname{cr}\left(K_{5, n}\right) \geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor \\
& \operatorname{cr}\left(K_{6, n}\right) \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor
\end{aligned}
$$

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From this he deduced that

$$
\operatorname{cr}\left(K_{m, n}\right) \geq \frac{1}{5} m(m-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor
$$

## Theorem2:

For $\mathrm{m}>\mathrm{n}, \operatorname{cr}(Z(m, n)) \geq c r\left(K_{m, n}\right) \leq c r\left(K_{n, m}\right)$.
Proof:
From theorem 1,

$$
\operatorname{cr}\left(K_{m, n}\right) \geq \frac{1}{5} m(m-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor
$$

By definition,

$$
\left.\mathrm{Z}(\mathrm{~m}, \mathrm{n})=\left\lfloor\frac{m}{2}\right\rfloor \frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor
$$

We can prove the theorem byinduction. Since in $\operatorname{cr}\left(K_{m, n}\right)$, there are ${ }^{m} c_{5} K_{5, n}$ subgraphs of $K_{m, n}$ with the partite with ' n ' vertices in $K_{5, n}$. So we shall obtain the lower bound of $c r\left(K_{m, n}\right)$ for $\mathrm{m} \geq 5$ and $\mathrm{n} \geq 3$.
Case(i): Let $\mathrm{n}=3$.
Subcase(i): $m=5$,
$\operatorname{cr}(Z(5,3))=2 \cdot 2 \cdot 1 \cdot 1=4$
$\operatorname{cr}\left(K_{5,3}\right)=\frac{1}{5} \cdot 5 \cdot 4 \cdot 1 \cdot 1=4$
$\operatorname{cr}\left(K_{3,5}\right)=\frac{1}{5} \cdot 3 \cdot 2 \cdot 2 \cdot 2=\frac{24}{5}=4.8$
$\therefore \operatorname{cr}(Z(5,3)) \geq \operatorname{cr}\left(K_{5,3}\right) \leq \operatorname{cr}\left(K_{3,5}\right)$
Subcase(ii): $\mathrm{m}=6$,
$\operatorname{cr}(Z(6,3))=3 \cdot 2 \cdot 1 \cdot 1=6$
$\operatorname{cr}\left(K_{6,3}\right)=\frac{1}{5} \cdot 6 \cdot 5 \cdot 1 \cdot 1=6$
$\operatorname{cr}\left(K_{3,6}\right)=\frac{1}{5} \cdot 3 \cdot 2 \cdot 3 \cdot 2=\frac{36}{5}=7 \cdot 2$
$\therefore \operatorname{cr}(Z(6,3)) \geq \operatorname{cr}\left(K_{6,3}\right) \leq \operatorname{cr}\left(K_{3,6}\right)$
Subcase(ii): $\mathrm{m}=7$,

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$\operatorname{cr}(Z(7,3))=3.3 .1 .1=9$
$\operatorname{cr}\left(K_{7,3}\right)=\frac{1}{5} \cdot 7 \cdot 6 \cdot 1 \cdot 1=\frac{42}{5}=8.4$
$\operatorname{cr}\left(K_{3,7}\right)=\frac{1}{5} \cdot 3 \cdot 2 \cdot 3 \cdot 3=\frac{54}{5}=10.8$
$\therefore \operatorname{cr}(Z(7,3)) \geq c r\left(K_{7,3}\right) \leq \operatorname{cr}\left(K_{3,7}\right)$
$\Rightarrow \operatorname{cr}(Z(m, 3)) \geq c r\left(K_{m, 3}\right) \leq c r\left(K_{3, m}\right)$
Case(ii): Let $\mathrm{n}=4$.
Subcase(i): $m=5$,

$$
\operatorname{cr}(Z(5,4))=2 \cdot 2 \cdot 2 \cdot 1=8
$$

$$
\operatorname{cr}\left(K_{5,4}\right)=\frac{1}{5} \cdot 5 \cdot 4 \cdot 2 \cdot 1=8
$$

$$
\operatorname{cr}\left(K_{4,5}\right)=\frac{1}{5} \cdot 4 \cdot 3 \cdot 2 \cdot 2=\frac{48}{5}=9.8
$$

$$
\therefore \operatorname{cr}(Z(5,4)) \geq \operatorname{cr}\left(K_{5,4}\right) \leq \operatorname{cr}\left(K_{4,5}\right)
$$

Subcase(ii): $m=6$,

$$
\begin{aligned}
& \operatorname{cr}(Z(6,4))=3 \cdot 2 \cdot 2 \cdot 1=12 \\
& \operatorname{cr}\left(K_{6,4}\right)=\frac{1}{5} \cdot 6 \cdot 5 \cdot 2 \cdot 1=12 \\
& \operatorname{cr}\left(K_{4,6}\right)=\frac{1}{5} \cdot 4 \cdot 3 \cdot 3 \cdot 2=\frac{72}{5}=14 \cdot 4 \\
& \therefore \operatorname{cr}(Z(6,4)) \geq \operatorname{cr}\left(K_{6,4}\right) \leq \operatorname{cr}\left(K_{4,6}\right)
\end{aligned}
$$

Subcase(ii): $\mathrm{m}=7$,

$$
\begin{aligned}
& \operatorname{cr}(Z(7,4))=3 \cdot 3 \cdot 2 \cdot 1=18 \\
& \operatorname{cr}\left(K_{7,4}\right)=\frac{1}{5} \cdot 7 \cdot 6 \cdot 2 \cdot 1=\frac{84}{5}=16.8 \\
& \operatorname{cr}\left(K_{4,7}\right)=\frac{1}{5} \cdot 4 \cdot 3 \cdot 3 \cdot 3=\frac{108}{5}=21.6 \\
& \therefore \operatorname{cr}(Z(7,4)) \geq \operatorname{cr}\left(K_{7,4}\right) \leq \operatorname{cr}\left(K_{4,7}\right) \\
& \Rightarrow \operatorname{cr}(Z(m, 4)) \geq \operatorname{cr}\left(K_{m, 4}\right) \leq \operatorname{cr}\left(K_{4, m}\right)
\end{aligned}
$$

In general,
$\operatorname{cr}(Z(m, n)) \geq \operatorname{cr}\left(K_{m, n}\right) \leq \operatorname{cr}\left(K_{n, m}\right)$.
We also observe that the following inequality,
$2 c r(Z(m, n+1)) \geq 2 c r\left(K_{m, n+1}\right) \leq 2 \operatorname{cr}\left(K_{n+1, m}\right)$
Also holds good for the above cases.

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Hence the proof.

## Theorem 2:

For $\mathrm{m}=\mathrm{n}, \operatorname{cr}(Z(m, n)) \geq \operatorname{cr}\left(K_{m, n}\right)$.That is,
$\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^{2}\left[\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)+\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right]=$
$\left.\left\lfloor\frac{n}{2}\right\rfloor^{4} \geq \frac{1}{5} n(n-1)\left\lfloor\frac{n}{2}\right\rfloor \frac{n-1}{2}\right\rfloor$
Proof:
When $\mathrm{m}=\mathrm{n}, Z(n, n)$ is a complete bipartite graph.
By theorem 1,

$$
\operatorname{cr}\left(K_{m, n}\right) \geq \frac{1}{5} n(n-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor
$$

We shall prove the theorem for large sufficiently larger ' $n$ ' and hence deducing the result for subsequent small ' $n$ '. Case(i):

$$
\begin{aligned}
& \operatorname{cr}\left(K_{11,11}\right)=625 \\
& =\frac{1}{2}\left[\begin{array}{l}
5(5)^{2}+4(5)^{2}+4(5)^{2}+4(5)^{2}+4(5)^{2}+4(5)^{2} \\
+5(5)^{2}+5(5)^{2}+5(5)^{2}+5(5)^{2}+5(5)^{2}
\end{array}\right] \\
& =\frac{1}{2}(5)^{2}[6.5+5.4] \\
& =\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^{2}\left[\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)+\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right] \\
& =\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^{3}\left[2\left\lfloor\frac{n}{2}\right\rfloor\right] \\
& =\left\lfloor\frac{n}{2}\right\rfloor^{4} \\
& \operatorname{cr}(Z(11,11))=550=\frac{1}{5} \cdot 11 \cdot 10 \cdot 5 \cdot 5 \\
& =\frac{1}{5} \cdot 11(11-1)\left\lfloor\frac{11}{2}\right\rfloor\left\lfloor\frac{11-1}{2}\right\rfloor \\
& =\frac{1}{5} \cdot n(n-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor \\
& \Rightarrow \operatorname{cr}(Z(11,11)) \geq \operatorname{cr}\left(K_{11,11}\right)
\end{aligned}
$$

Case(ii):

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$$
\begin{aligned}
\operatorname{cr}(Z(9,9)) & =256=\frac{1}{5} \cdot 9 \cdot 8 \cdot 4 \cdot 4 \\
& =\frac{1}{5} \cdot 9(9-1)\left\lfloor\frac{9}{2}\right\rfloor\left\lfloor\frac{9-1}{2}\right\rfloor \\
& =\frac{1}{5} \cdot n(n-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor \\
\operatorname{cr}\left(K_{9,9}\right)= & 256 \\
= & \frac{1}{2}\left[4(4)^{2}+3(4)^{2}+3(4)^{2}+3(4)^{2}+3(4)^{2}+4(4)^{2}+4(4)^{2}\right] \\
= & \frac{1}{2}(4)^{2}[5 \cdot 4+4 \cdot 3] \\
= & \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^{2}\left[\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)+\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right] \Rightarrow \operatorname{cr}(Z(9,9)) \geq \operatorname{cr}\left(K_{9,9}\right) \\
= & \frac{1}{2}\left\lfloor\frac{n}{2}\right]^{3}\left[2\left\lfloor\frac{n}{2}\right\rfloor\right] \\
= & \left\lfloor\frac{n}{2}\right\rfloor^{4}
\end{aligned}
$$

In general,

$$
\operatorname{cr}(Z(m, n)) \geq \operatorname{cr}\left(K_{m, n}\right)
$$

Hence the proof.

## III. CONCLUSION

We have given an alternate way of finding crossings in complete bipartite graphs. We also proved in bipartite graphs, the best lower bound of $\operatorname{cr}\left(K_{m, n}\right)$ will always be a lower bound until ' m ' and ' n ' are altered.

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