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An Advance Class of Analytic Functions with Fekete-Szegö Inequality using subordination Method

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Abstract: In this Paper we have introduced an advance class of analytic functions along with its subclasses by using principle of subordination and as so obtained sharp upper Bound of the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to these classes.

Extremal functions are also investigated.

Keywords: Bounded functions, Close to convex function, extremal function, Inverse Starlike functions, Starlike functions, Univalent functions.

I. INTRODUCTION

Let \mathcal{A} denote the class of analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

In the unit disc $= \{z: |z| < 1\}$, \mathcal{S} be the class of analytic univalent functions in \mathbb{E} .

Bieber Bach [7] proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. and Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the above known estimates this inequality plays an important role to determining estimates of higher coefficients for some sub classes \mathcal{S} {Chhichra [11], Babalola [6]}

Using Löwner's method [5] In 1933, Fekete and szego investigated a well known relation between a_3 and a_2^2 for the class \mathcal{S}

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & , \text{if } \mu \leq 0 \\ 1 + 2e^{\frac{-2\mu}{1-\mu}} & , \text{if } 0 \leq \mu \leq 1 \\ 4\mu - 3 & , \text{if } \mu \geq 1 \end{cases} \tag{1.2}$$

Let us define some subclasses of \mathcal{S}

Let \mathcal{K} denotes subclasses of \mathcal{S} of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$ satisfying the condition

$$Re \frac{(zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.3}$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exist $g(z) \in \mathcal{S}^*$ such that

$$Re \frac{(zf'(z))}{g(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

The class of close to convex functions introduced by Kaplan [17], and he proved that close to convex functions are univalent.

$$\mathcal{S}^*(A,B) = \{ f(z) \in \mathcal{A} ; \frac{(zf'(z))}{g(z)} < \frac{1+Az}{1+Bz}, -1 \leq B \leq A \leq 1, z \in \mathbb{E} \} \tag{1.5}$$

Where $\mathcal{S}^*(A,B)$ is a subclass of \mathcal{S}^* .

For strongly alpha quasi-convex functions Fekete-Szegö problem was studied by Abdel-Gawad [3]. The upper bound of $|a_3 - \mu a_2^2|$ for different functions in the class \mathcal{S} has been investigated by many authors including Goel and Mehrook [13] and recently by Al-Shaqsi and Darus [4] Hayami and Owa [16], Al-Abadi and Darus [10].

Gurmeet singh et al. [3] introduced the class of inverse Starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$Re \left(\frac{zf(z)}{2 \int_0^z f(z) dz} \right) > 0, z \in E \quad \text{i.e.} \quad \frac{zf(z)}{2 \int_0^z f(z) dz} < \frac{1+z}{1-z} \tag{1.6}$$

Gandhi et al. [14] established a new class of analytic functions with Fekete-szego inequality using subordination method.

Here introduce the class \mathcal{A} , of Univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$\left[\frac{z\{zf(z)\}'}{2f(z)} \right] < \left(\frac{1+z}{1-z} \right)^\alpha ; \alpha > 0 \tag{1.7}$$

And subclass consisting of the functions $g(z) = z + \sum_{n=2}^\infty b_n z^n \in \mathcal{A}$ satisfying the condition

$$\left[\frac{z\{zf(z)\}'}{2f(z)} \right] < \left(\frac{1+Az}{1+Bz} \right)^\alpha ; -1 \leq B \leq A \leq 1 ; \alpha > 0 \tag{1.8}$$

Here, Symbol $<$ stands for subordination, defined as follows:

A. Principle of Subordination

If $f(z)$ and $F(z)$ are two functions which are analytic in \mathbb{E} , then $f(z)$ is called a subordinate to $F(z)$ in \mathbb{E} , if there exists a function $w(z)$ which is analytic in \mathbb{E} satisfying the conditions

(i) $w(0) = 0$ and (ii) $|w(z)| < 1$

such that $f(z) = F(w(z))$, where $z \in \mathbb{E}$ and we denote it as $f(z) < F(z)$.

Let \mathcal{U} denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^\infty d_n z^n, w(0) = 0, |w(z)| < 1 \tag{1.9}$$

With $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$.

II. RESULTS AND DISCUSSION

1) *Theorem 1:* If $f(z) \in \mathcal{A}$, then the result

$$|a_3 - \mu a_2^2| \leq \begin{cases} 10\alpha^2 - 16\mu\alpha^2 & , \text{if } \mu \leq \frac{5\alpha-1}{8\alpha} \end{cases} \tag{1.10}$$

$$2\alpha & , \text{if } \frac{5\alpha-1}{8\alpha} \leq \mu \leq \frac{5\alpha+1}{8\alpha} \tag{1.11}$$

$$16\mu\alpha^2 - 10\alpha^2 & , \text{if } \mu \geq \frac{5\alpha+1}{8\alpha} \tag{1.12}$$

is sharp.

Proof: By using expansion method (1.7) leads to

$$1 + \frac{1}{2} a_2 z + (a_3 - \frac{1}{2} a_2^2) z^2 + \dots = 1 + 2\alpha c_1 z + 2\alpha(c_2 + \alpha c_1^2) z^2 + \dots \tag{1.13}$$

After Identifying the terms we have

$$|a_3 - \mu a_2^2| \leq |2\alpha c_2 + 10\alpha^2 c_1^2 - 16\mu\alpha^2 c_1^2|$$

This leads to

$$|a_3 - \mu a_2^2| \leq 2\alpha + [|10\alpha^2 - 16\mu\alpha^2| - 2\alpha] |c_1|^2 \tag{1.14}$$

Case I: If $\mu \leq \frac{5}{8}$, then (1.14) leads to

$$|a_3 - \mu a_2^2| \leq 2\alpha + [(10\alpha^2 - 2\alpha) - 16\mu\alpha^2] |c_1|^2 \tag{1.15}$$

Subcase I(a): If $\mu \leq \frac{5\alpha-1}{8\alpha}$, then (1.15) leads to

$$|a_3 - \mu a_2^2| \leq 10\alpha^2 - 16\mu\alpha^2 \tag{1.16}$$

Subcase I(b): If $\mu \geq \frac{5\alpha-1}{8\alpha}$, then (1.15) leads to

$$|a_3 - \mu a_2^2| \leq 2\alpha \tag{1.17}$$

Case II: If $\mu \geq \frac{5}{8}$, then (1.14) leads to

$$|a_3 - \mu a_2^2| \leq 2\alpha + [16\mu\alpha^2 - (10\alpha^2 + 2\alpha)] |c_1|^2 \tag{1.18}$$

Subcase II(a): If $\mu \leq \frac{5\alpha+1}{8\alpha}$, then (1.18) leads to

$$|a_3 - \mu a_2^2| \leq 2\alpha \tag{1.19}$$

Subcase II(b): If $\mu \geq \frac{5\alpha+1}{8\alpha}$, then (1.18) leads to

$$|a_3 - \mu a_2^2| \leq 16\mu\alpha^2 - 10\alpha^2 \tag{1.20}$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \leq 2\alpha , \text{if } \frac{5\alpha-1}{8\alpha} \leq \mu \leq \frac{5\alpha+1}{8\alpha} \tag{1.21}$$

This completes the theorem, therefore the result is sharp.

Extremal function for the first and third inequality is given by

$$f_1(z) = \frac{z}{(1-\alpha z)^4} \tag{1.22a}$$

And Extremal function for the second inequality is given by

$$f_2(z) = \frac{z}{(1-\alpha z^2)^2} \tag{1.22b}$$

2) *Theorem 2* : If $f(z) \in \mathcal{A}$, then the result

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} (A-B)(2A-3B)\alpha^2 - 4\mu\alpha^2(A-B)^2 & , \text{if } \mu \leq \frac{(2A-3B)\alpha-1}{4(A-B)\alpha} \end{cases} \tag{1.23a}$$

$$\begin{cases} (A-B)\alpha & , \text{if } \frac{(2A-3B)\alpha-1}{4(A-B)\alpha} \leq \mu \leq \frac{(2A-3B)\alpha+1}{4(A-B)\alpha} \end{cases} \tag{1.23b}$$

$$\begin{cases} 4\mu\alpha^2(A-B)^2 - (A-B)(2A-3B)\alpha^2 & , \text{if } \mu \leq \frac{(2A-3B)\alpha+1}{4(A-B)\alpha} \end{cases} \tag{1.23c}$$

is sharp.

Proof: By using expansion method (1.8) leads to

$$1 + \frac{1}{2}a_2 z + (a_3 - \frac{1}{2}a_2^2) z^2 + \dots = 1 + (A-B)\alpha c_1 z + (A-B)\alpha(c_2 - B\alpha c_1^2) z^2 + \dots \tag{1.24}$$

After Identifying the terms in (1.24) we have

$$|a_3 - \mu a_2^2| \leq |(A-B)\alpha(c_2 - B\alpha c_1^2) + 2(A-B)^2\alpha^2 c_1^2 - 4\mu(A-B)^2\alpha^2 c_1^2|$$

This leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha + \{|2(A-B)^2\alpha^2 - B(A-B)\alpha^2 - 4\mu(A-B)^2\alpha^2| - (A-B)\alpha\} |c_1|^2 \tag{1.25}$$

here two cases arise:

Case I: If $\mu \leq \frac{2A-3B}{4(A-B)}$, then (1.25) leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha + [(A-B)\{(2A-3B)\alpha-1\}\alpha - 4\mu(A-B)^2\alpha^2] |c_1|^2 \tag{1.26}$$

Under this case (1.26) two subcases arise:

Subcase I(a): If $\mu \leq \frac{(2A-3B)\alpha-1}{4(A-B)\alpha}$, then (1.26) leads to

$$|a_3 - \mu a_2^2| \leq \{(A-B)(2A-3B)\alpha - 4\mu(A-B)^2\alpha^2\} |c_1|^2 \tag{1.27}$$

Subcase I(b): If $\mu \geq \frac{(2A-3B)\alpha-1}{4(A-B)\alpha}$, then (1.26) leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha \tag{1.28}$$

Case II: If $\mu \geq \frac{2A-3B}{4(A-B)}$, then (1.25) leads to

$$|a_3 - \mu a_2^2| \leq (A-B) + [4\mu(A-B)^2\alpha^2 - (A-B)\{(2A-3B)\alpha+1\}\alpha] |c_1|^2 \tag{1.29}$$

Under this case again two subcases arise:

Subcase II(a): If $\mu \leq \frac{(2A-3B)\alpha+1}{4(A-B)\alpha}$, then (1.29) leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha \tag{1.30}$$

Subcase II(B): $\mu \geq \frac{(2A-3B)\alpha+1}{4(A-B)\alpha}$, then (1.29) leads to

$$|a_3 - \mu a_2^2| \leq \{4\mu(A-B)^2\alpha^2 - (A-B)(2A-3B)\alpha^2\} |c_1|^2 \tag{1.31}$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha, \text{ if } \frac{(2A-3B)\alpha-1}{4(A-B)\alpha} \leq \mu \leq \frac{(2A-3B)\alpha+1}{4(A-B)\alpha} \tag{1.32}$$

This completes the theorem therefore, the result is sharp.

Extremal function for the first and third inequality is given by

$$f_1(z) = z \left\{ 1 + \left(\frac{2A-2B}{2A-3B} \right) z\alpha \right\}^{2A-3B} \tag{1.33a}$$

And for the for second inequality is given by

$$f_2(z) = \frac{z}{(1-\alpha z^2)^{A-B}} \tag{1.33b}$$

III. CONCLUDING REMARKS AND COROLLARIES

1) *Corollary 1.1:* Taking $\alpha = 1$ in the theorem 1, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 10 - 16\mu & , \text{if } \mu \leq \frac{1}{2} \end{cases} \quad (1.34)$$

$$\begin{cases} 2 & , \text{if } \frac{1}{2} \leq \mu \leq \frac{3}{4} \end{cases} \quad (1.35)$$

$$\begin{cases} 16\mu - 10 & , \text{if } \mu \geq \frac{3}{4} \end{cases} \quad (1.36)$$

These estimates were derived by Keogh and Merkes [1] and the results are for the class of univalent convex functions.

Further if we take $A = 1$ and $B = -1$ ($-1 \leq B \leq A \leq 1$) in the result of theorem 2, we get the result of theorem 1, therefore our result for the theorem 2 reduces to the result of the theorem 1. Hence theorem 2 is the generalization of theorem 1. And the results are sharp and also if we put $A = 1$ and $B = -1$ in extremal function of theorem 2, we get the extremal function of theorem 1. The extremal function given by [(1.22a),(1.22b)] increases as α increases and decreases as α decreases and the extremal function given by [(1.33a),(1.33b)] also increases and decreases as α increases and decreases respectively. Hence extremal function is an increasing function.

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