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Virtual Principal of Solid Mechanics

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Abstract: The main objective of solid mechanics is to find deformation, deflection, and stress measurements for continuum bodies.
Here I adopted 4 types of principals to analyze trusses and beam. As we know in engineering solid mechanics is the fundamental base for civil, aerospace, nuclear, biomedical, mechanical, material science, physics, and many subsidiary branches respectively. However mathematical formulation for many application-based software developments is foremost. So, for validation purposes, I discuss linear statically indeterminate structure examples subsequently.
Keywords: Virtual Work, Complementary Virtual Work, Stationary Potential Energy, Stationary Complementary Energy

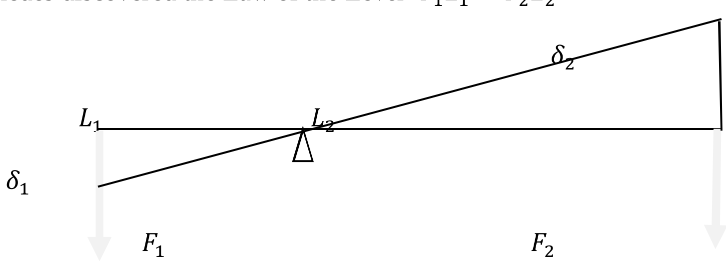
I. INTRODUCTION

My analysis is depending upon detailed knowledge of applied mechanics and mathematics. Contemporary conditions of equilibrium or when it is not changing rotational or translation motion are considered to determine the shape of solid bodies and stress generation under the application of forces. In structural designing deflection and stress play a major role which is always needed under permissible limits. For determining stress-strain the objectives of the paper are

A. Familiarity with the concept of virtual work
B. Familiarity with the concept of complementary virtual work
C. Familiarity with the concept of stationary potential energy
D. Familiarity with the concept of stationary complementary energy

II. THE CONCEPT OF VIRTUAL WORK

The first step towards the concept of an optimum path followed by nature: Archimedes (287-212BC) was a Greek mathematician who used the concept of virtual work and infinitesimal calculus for equilibrium-for the first time in history!
Archimedes discovered the Law of the Lever $F_1L_1 = F_2L_2$



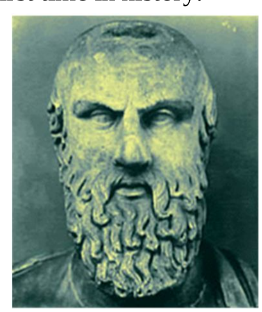


Figure 1. The lever displaced by the admissible displacement

He employed the following simple rule for equilibrium. Virtual (imaginary and geometrically compatible/admissible) displacements δ_1 and δ_2 are imposed upon the lever system, already in equilibrium, as shown in Figure 1.

Geometric Compatibility: $\frac{\delta_1}{L_1} = \frac{\delta_2}{L_2}$ (1)

He then stated that the condition for the equilibrium of the lever is that the net virtual work done by the forces moving over the virtual displacement should vanish;

Equilibrium: $F_1\delta_1 - F_2\delta_2 = 0$ (2)

Combining (1) and (2), one gets the lever rule

$$F_1L_1 = F_2L_2 \quad (3)$$

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Virtual work is the total work done by the applied forces and the inertial forces of a mechanical system as it moves through a set of virtual displacements. For a body to be in static equilibrium, the principle of least action requires that the virtual work done by the applied external forces and the internal resistive forces over any admissible virtual displacement field to be zero.

A. The principle of Virtual Work

For any arbitrary yet admissible set of virtual displacements, superposed over the equilibrium configuration of a structural system, the net virtual work done by the forces (external and internal) is zero.

i.e. Equilibrium ↔ Zero virtual work

Or $$\delta W_{ext} = \delta W_{int} = 0 \quad (4)$$

Virtual work done by external forces = Virtual work done by internal forces

1) *Example 1:* Using the virtual work principle, derive the equations of equilibrium for the elastic bar system loaded vertically by a load P as shown in Figure 2. Given that Young's Modulus *E*=200 KN /mm^2 and the vertical load is *P*=40 KN, calculate the vertical and horizontal of the joint B.

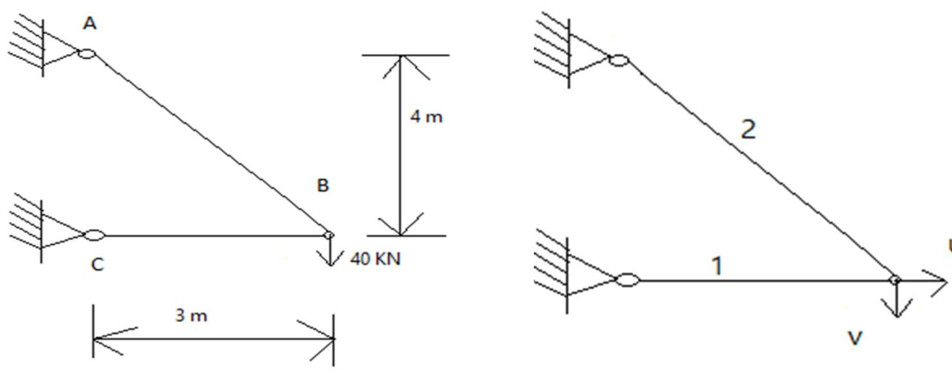

Figure 2. A simple truss loaded by a tip vertical load.

TABLE I

Bar(i)	Member 1	Member 2
Length(mm)	3000	5000
Elasticity rigidity(KN/mm^2)	200	200
Area(mm^2)	100	100
The angle of inclination θ with global +X axis (α)	0°	126.87°

Properties of the members of the truss of example 1

Let
u = *x* component of displacement at point B
v = *y* component of displacement at point B
α_i= Angle of the i^{th} member with respect to +*x* axis (*i*=1, 2)
L_i=Original length of the i^{th} member (*i*=1, 2)
Actual strain (at equilibrium) in the i^{th} bar is

$$\varepsilon_i=\frac{u.cos\alpha_i}{L_i}+\frac{v.cos\alpha_i}{L_i} \quad (a)$$

Actual stress (at equilibrium) in the i^{th} bar is

$$\sigma_i = E_i\varepsilon_i = E_i\left(\frac{u.cos\alpha_i}{L_i}+\frac{v.sin\alpha_i}{L_i}\right) \quad (b)$$

Here E_i is Young's Modulus of the i^{th} bar.

Apply a virtual displacement δu along direction u. By the principle of virtual work,

$$\delta W_{ext} = \delta W_{int} \qquad (c)$$

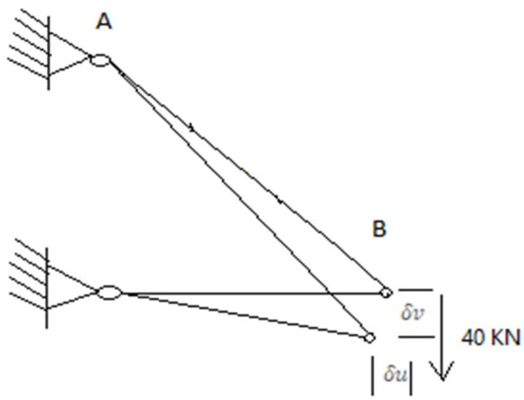

Figure 3. Virtual displacements imposed upon point B.

$$\delta u.0 = \sum_{i=1}^{3} \sigma_i \delta\varepsilon_i.(A_i L_i) = \sum_{i=1}^{3} E\left(\frac{ucos\alpha_i}{L_i} + \frac{vsin\alpha_i}{L_i}\right)\left(\frac{\delta u.cos\alpha_i}{L_i}\right)(A_i L_i)$$

$$0 = \sum_{i=1}^{3} \frac{EA_i}{L_i}(ucos^2\alpha_i + vsin\alpha_i cos\alpha_i) \qquad (d)$$

Apply a virtual displacement δv along with direction v. Again, by the principle of virtual work,

$$\delta v.P = \sum_{i=1}^{3} \sigma_i \delta\varepsilon_i.(A_i L_i) = \sum_{i=1}^{3} E\left(\frac{u.cos\alpha_i}{L_i} + \frac{v.sin\alpha_i}{L_i}\right)\left(\frac{\delta v.cos\alpha_i}{L_i}\right)(A_i L_i)$$

Or $P = \sum_{i=1}^{3} \frac{EA_i}{L_i}(u.sin\alpha_i cos\alpha_i + v.sin^2\alpha_i)$ (e)

Combining equations (d) and (e), the equation of equilibrium for the bar system is

$$\begin{bmatrix} \sum_{i=1}^{3} \frac{E_i A_i}{L_i}(cos^2\alpha_i) & \sum_{i=1}^{3} \frac{E_i A_i}{L_i}(sin\alpha_i.cos\alpha_i) \\ \sum_{i=1}^{3} \frac{E_i A_i}{L_i}(sin\alpha_i.cos\alpha_i) & \sum_{i=1}^{3} \frac{E_i A_i}{L_i}(sin^2\alpha_i) \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \qquad (f)$$

Using the given values, one gets

$$10 * \begin{bmatrix} \frac{2}{3} * 1 + \frac{2}{5} * .36 & \frac{2}{3} * 0 + \frac{2}{5} * .480050 \\ \frac{2}{3} * 0 + \frac{2}{5} * .480050 & \frac{2}{3} * 0 + \frac{2}{5} * .6399 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \end{bmatrix}$$

Solving this equation gives the displacement components *u* and *v*, from which the stresses in the bars can be determined from equation (*b*).

u =4.5mm (left)

v = 19 mm (down)

i.e. Joint B moves 4.5 mm to the left & 19 mm downward. The internal stresses in the bars are calculated as

$$\sigma_{AB} = 500\ N/mm^2$$

$$\sigma_{BC} = -300\ N/mm^2$$

The internal forces (stress resultants) in the bars are given by

$F_{AB} = \sigma_{AB}.A_{AB} = 50KN$ (Tension)

$F_{BC} = \sigma_{BC}.A_{BC} = -30KN$ (Compression)

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III. THE CONCEPT OF COMPLEMENTARY VIRTUAL WORK

In indeterminate, or over-constrained structural systems (wherein there is more number of members or supports than the minimum required for stability), the equations of static equilibrium alone cannot yield the solutions. For such systems, the objective is to extract the actual system of forces in equilibrium (of all the possible equilibrium configurations) that is compatible with the system constraints. Thus through the principle of complementary virtual work, one imposes a variation of the forces in equilibrium to search out the compatible one among all the possible equilibria.

A. The Principle Of Complementary Virtual Work

For the deformation of the system to be compatible with any self-equilibrating virtual force system, the external complementary virtual work is equal to internal complementary virtual work.

We have;

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$\sum y_i \,.\, \delta F_i = \sum e_i . \delta P_i \qquad (5)$$

Wherein (y_i, e_i) set of compatible displacement and (F_i, P_i) set of forces in equilibrium.

1) *Example 2:* A cantilever carrying a uniformly distributed load intensity *w* per unit length is propped at its free end such that it is at the level of the fixed end. What is the reaction force at the prop?

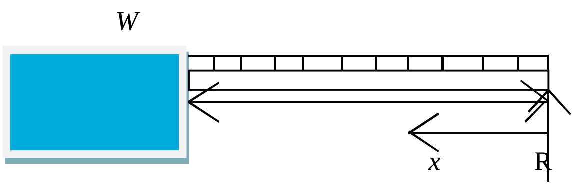

Figure 4. Simply supported Cantilever beam

Let *R* = Reaction force at the prop and *M* = Bending Moment distribution under equilibrium conditions. Then

$$M = \frac{Wx^2}{2} - Rx \qquad (a)$$

For a linearly elastic structure, the elastic strain energy U is equal to its complementary energy U^* for the beam,

Complementary energy: $U^* = U = \int \frac{M^2 dx}{2EI}$ $\qquad (b)$

Now a virtual variation δR in the prop reaction brings about a virtual change δM in the bending moment, given by

$$\delta M = -\delta R . x \qquad (c)$$

Thus the virtual change in complementary energy is given by the virtual complementary work,

$$\delta U^* = \int \frac{2M\delta M}{2EI} dx = \int_0^L \frac{M}{EI} \delta M dx = \int_0^L \frac{\left(\frac{wx^2}{2} - Rx\right)}{EI} \{-\delta R . x\} dx$$

$$= \frac{\delta R}{EI} L^3 \left\{-\frac{wL}{8} + \frac{R}{3}\right\} \qquad (d)$$

By the principle of complementary virtual work,

$$\delta U^* = \delta W^*$$

Or

$$\frac{\delta R}{EI} L^3 \left\{-\frac{wL}{8} + \frac{R}{3}\right\} = \delta R * 0$$

$$\frac{R}{3} = \frac{wL}{8}$$

$$R = \frac{3wL}{8} \qquad (e)$$

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TABLE II.

Principle of virtual work (displacement method for finding equilibrium from all possible admissible /compatible virtual displacement fields)	Principle of complementary virtual work (force method for finding geometrically compatible force fields from all possible virtual variations of forces in equilibrium)
$\delta W_E = \delta U$	$\delta W_E^* = \delta U^*$
$\delta W_E = \sum_{i=1}^{n} Q_i \delta q_i$ $\delta U = \int \sigma_{ij} \delta e_{ij} dv$	$\delta W_E^* = \sum_{i=1}^{n} \delta Q_i$ $\delta U^* = \int \sigma_{ij} \delta e_{ij} dv$

IV. THE CONCEPT OF STATIONARY POTENTIAL ENERGY

The potential energy of a system is its total energy that comprises of the strain energy (internal work) and the potential of the externally applied forces (negative of the external work done by the forces). The principle of stationary potential energy extracts the geometrically admissible displacement field that corresponds to equilibrium conditions.

A. *The Principle Of Stationary Total Potential Energy*

"Of all the possible admissible displacements (satisfying the kinematics boundary conditions) the one that corresponds to equilibrium (of forces) is the one that makes the potential energy stationary."

B. *Total Potential Energy*

Total Potential Energy = Strain Energy – Work done by external forces
Or if

$$\Pi = U - W$$

Equilibrium Condition: For a virtual displacement field δu

$$\delta\Pi = 0 \qquad (6)$$

i.e. Any virtual admissible displacement δu about equilibrium (of forces) brings about a vanishing first variation of the total potential energy. Equilibrium corresponds to the admissible virtual displacement which makes the total potential energy Π a stationary one. A stable equilibrium makes the total potential energy a minimum.

1) *Example 3:* Using the principle of stationary total potential energy, derive the equations of equilibrium for the elastic bar system loaded vertically by a load P as shown in Figure 5 (same problem for example 1). Given that Young's Modulus E=200 KN $/mm^2$ and the vertical load is P=40 KN, calculate the vertical and horizontal of the joint B.

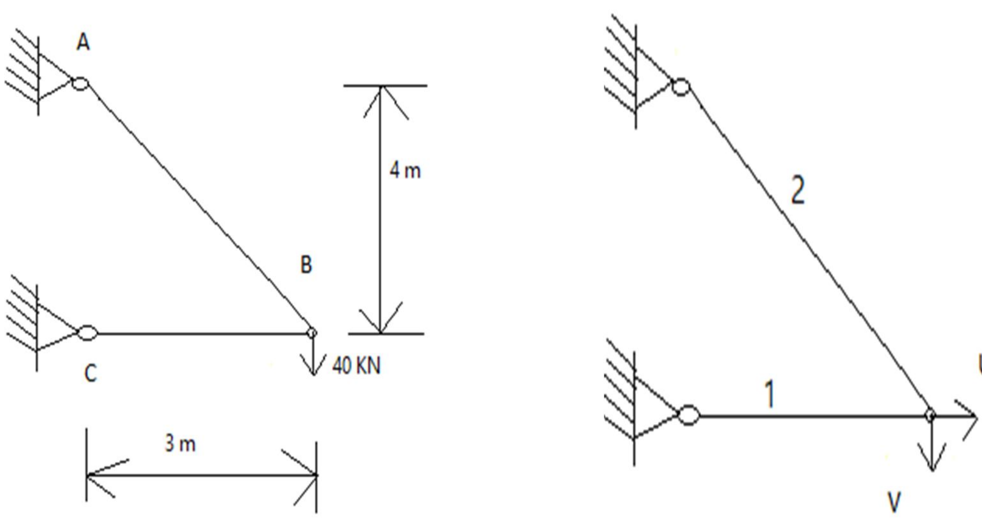

Figure 5. A simple truss loaded by a tip vertical load

Actual strain (at equilibrium) in the i^{th} bar is

$$\varepsilon_i = \frac{u.cos\alpha_i}{L_i} + \frac{v.cos\alpha_i}{L_i} \qquad (a)$$

Actual stresses in the i^{th} bar is

$$\sigma_i = E_i\varepsilon_i = E_i\left(\frac{u.cos\alpha_i}{L_i} + \frac{v.sin\alpha_i}{L_i}\right) \qquad (b)$$

Here E_i is the Young's Modulus of the i^{th} bar.

Strain Energy:

$$U = \frac{1}{2}\sum_{i=1}^{2} E_i\,\varepsilon_i^2(Volume_i) = \frac{1}{2}\sum_{i=1}^{2} E_i\,\varepsilon_i^2(A_iL_i) \qquad (c)$$

$$U = \frac{1}{2}\sum_{i=1}^{3}\frac{E_iL_i}{L_i}(u.cos\alpha_i + v.sin\alpha_i)^2 \qquad (d)$$

Work done by external vertical load P acting at B:

$$W = P.v \qquad (e)$$

Total potential energy of the system:

$$\prod = U - W = \frac{1}{2}\sum_{i=1}^{2}\frac{E_iA_i}{L_i}(u.\cos\alpha_i + v.\sin\alpha_i)^2 - \{(0.u + P.v)\} \qquad (f)$$

At equilibrium, the first variation of the potential energy should vanish:

i.e. $\delta\Pi = \frac{\partial\Pi}{\partial u}\delta u + \frac{\partial\Pi}{\partial v}\delta v = 0$ For non-trivial $\delta u, \delta v$ (g)

Equilibrium: $\frac{\partial\Pi}{\partial u} = 0 \quad \frac{\partial\Pi}{\partial v} = 0$

$$\Pi = \frac{1}{2}\sum_{i=1}^{2}\frac{E_iA_i}{L_i}(u.\cos\alpha_i + v.\sin\alpha_i)^2 - P.v$$

Equilibrium: $\frac{\partial\Pi}{\partial u} = 0 \quad \frac{\partial\Pi}{\partial v} = 0$

$$0 = \sum_{i=1}^{2}\frac{E_iA_i}{L_i}(u.\cos^2\alpha_i + v.\sin\alpha_i\cos\alpha_i) \qquad (i)$$

$$P = \sum_{i=1}^{2}\frac{E_iA_i}{L_i}(u.\sin\alpha_i\cos\alpha_i + v.\sin^2\alpha_i) \qquad (j)$$

The equation of equilibrium can now be expressed as

$$[K]\{d\} = \{F\}$$

$$\begin{bmatrix} \sum_{i=1}^{2}\frac{E_iL_i}{L_i}(\cos^2\alpha_i) & \sum_{i=1}^{2}\frac{E_iL_i}{L_i}(\sin\alpha_i.\cos\alpha_i) \\ \sum_{i=1}^{2}\frac{E_iL_i}{L_i}(\sin\alpha_i.\cos\alpha_i) & \sum_{i=1}^{2}\frac{E_iL_i}{L_i}(\sin^2\alpha_i) \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

Using the given values, one gets

$$10 * \begin{bmatrix} \frac{2}{3}*1 + \frac{2}{5}*.36 & \frac{2}{3}*0 + \frac{2}{5}*.480050 \\ \frac{2}{3}*0 + \frac{2}{5}*.480050 & \frac{2}{3}*0 + \frac{2}{5}*.6399 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \end{bmatrix}$$

u= 4.5mm (left)

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i.e. Joint B moves 4.5 mm to the left & 19 mm downward. The internal stresses in the bars are calculated as

$$\sigma_{AB} = 500\ N/mm^2$$

$$\sigma_{BC} = -300\ N/mm^2$$

$$F_{AB} = \sigma_{AB}.A_{AB} = 50KN \quad \text{(Tension)}$$

$$F_{BC} = \sigma_{BC}.A_{BC} = -30KN \quad \text{(Compression)}$$

V. THE CONCEPT OF STATIONARY COMPLEMENTARY ENERGY

For an elastic body in equilibrium under the action of applied forces, the internal forces (stresses) and reactions that are compatible with the kinematic constraints are those for which the total complementary energy has a stationary value. This concept is particularly useful in the solution of statically indeterminate structures.

A. *The Principle Of Stationary Complementary Energy*

"Of all possible admissible forces in equilibrium in a structural system, the one that corresponds to kinematic compatibility with the system constraints is the one that makes the total complementary energy stationary."

B. *Total Complementary Energy*

Total Complementary Energy = Complementary Strain Energy – Complementary Work

Or if

$$\Pi^* = U^* - W^*$$

Then Compatibility Condition: $\delta\Pi^* = 0$ (for δF) (*7*)

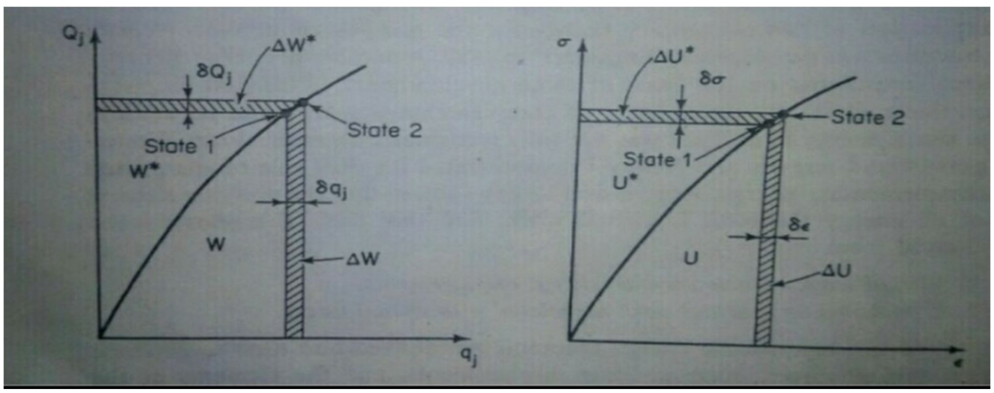

Figure 6. Complementary Energy & Strain Energy Diagram

W = Strain energy, W^* = Complementary Work
U = Strain energy, U^* = Complementary strain energy

i.e. Any virtual change of forces δQ in equilibrium corresponds to kinematic compatibility under the given constraints provided it brings about vanishing of the first variation of complementary energy Π^*.

1) *Example 4:* Determine the member forces of a loaded statically indeterminate plane truss (shown in Figure 7) by the principle of total complementary energy (force method).

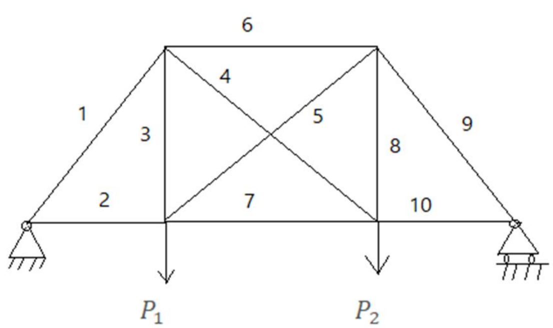

Figure 7. Plane Truss of Example 4 Degree of static indeterminacy of the truss

DI = m + r -2j =10+3 -2*6 =1 (*a*)

P_1 & P_2 are external loads.

a) *Step 1:* Let us convert the truss to a statically determinate primary structure by "cutting" the fifth member of the truss (Figure 8).

Using only the equations of static equilibrium it is possible to determine the internal member forces F_p of the primary structure. (p=1 2 3 …. 10 with $F_5 = 0$)

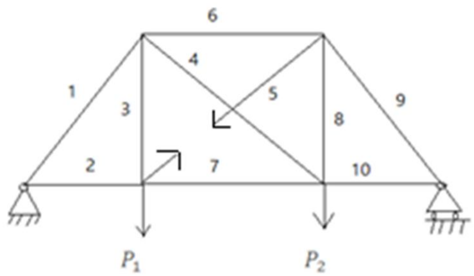

Figure. 8 Primary (statically determinate structure) obtained by removing one redundant member (member 5).

b) *Step 2:* Apply equal and opposite forces f_5 on the determinate primary structure at the cut along with the fifth member of the truss. Using only equations of static equilibrium, it is possible to determine the internal member forces f_P of the bar.

$f_p = C_p f_5$; $C_5 = 1$; p=1, 2 ….. 10 (*b*)

By superposition, net internal forces in the members of the indeterminate truss:

$$\overline{F_p} = F_p + C_p f_5 \quad \text{(c)}$$

(Internal forces in the bars with external loads)

Any value of the force f_5 can satisfy this equation of equilibrium. For its actual value, we use the Principle of Stationary Complementary energy (compatibility),

$$\delta\Pi^* = \delta U^* - (\delta f_5 . u_{cut}\ \delta f_5 . u_{cut}) = \frac{\partial U^*}{\partial f_5}\delta f_5 = 0 \quad (d)$$

$(\delta f_5 . u_{cut} - \delta f_5 . u_{cut})$ Equal and opposite forces at the cut do zero work.

$$\Rightarrow \frac{\partial U^*}{\delta f_5} = \frac{d}{d f_5}\sum_{p=1}^{10}\frac{1}{2}\frac{(F_p + C_p f_5)^2}{(EA)_R} L_p = 0 \quad (e)$$

This equation provides the actual value of the force f_5 compatible with the indeterminate system.

Net force in the *p* member: $S_p = F_p + C_p f_5$ (*f*)

p = 1, 2 …….. 10

TABLE III.

Principle of minimum potential energy	Principle of complementary virtual work
$\delta\Pi = \delta(U + V_E) = 0$	$\delta U^* = \delta(U^* + V_E^*) = 0$
$U = \int (\int_0^{e_{ij}} \sigma_{ij} de_{ij}) dv$	$U^* = \int (\int_0^{e_{ij}} e_{ij} d\sigma_{ij}) dv$
$V_E = -\sum_{i=1}^{n} Q_i q_i$	$V_E^* = -\sum_{i=1}^{n} Q_i q_i$

VI. CONCLUSION

So far, only 4 types of the principal have been analyzed. In general application, the finite element method is more superior methods to find out structural designing problems and many commercial or open-source software for structural analysis is incorporated with that. Anyone who has a basic knowledge of programming can easily solve similar problems with Python, Matlab, Scilab libraries, or similar types of software packages.

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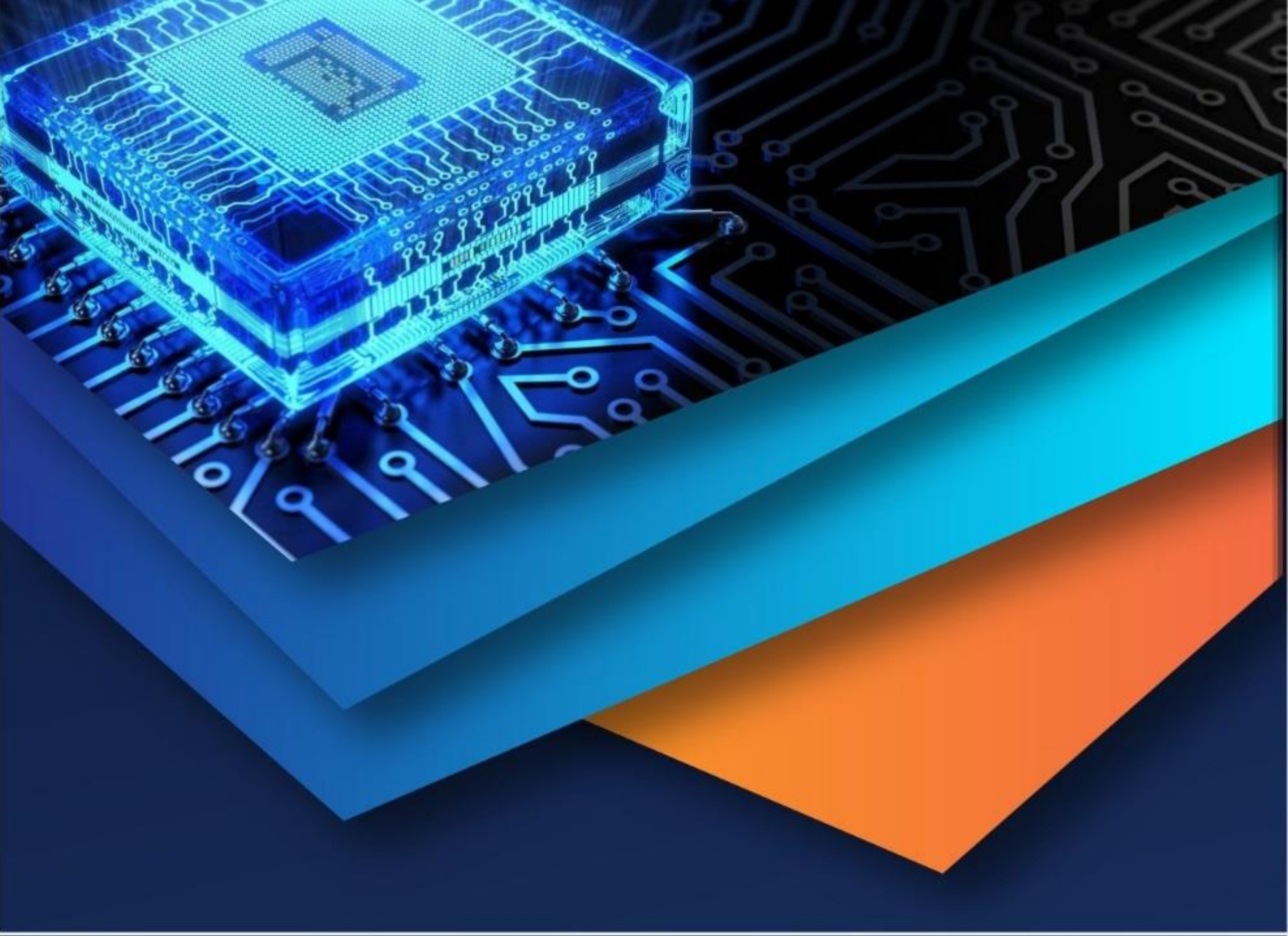

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