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An Overview of Generating Functions of Special Functions

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Abstract: Special functions are expressed as differential and integral representations of some elementary functions. This general representation leads to a unique and unified representation of special functions. The integral representation gives the simplified formulations rather than differential representation.
Keywords: Generating functions, Special functions,Integral representation, Schläfli integrals

I. INTRODUCTION

To solve the problems in theoretical and mathematical physics we many times encounter the differential equations[1]-[4]. The solutions of these equations are well known special functions of mathematical physics. Many times the problem of unification becomes an intelligent challenge for increasing number of functions. Unification of differential equations is, most of the times, governed by Sturm-Liouville theory[5]-[8]. Let the function p(x), q(x) and r(x) be analytic functions and λ an eigen value then the unified differential equation covering most of the special functions is :-

$$\frac{\partial}{\partial x}\left\{r(x)\frac{\partial y}{\partial x}\right\}+\left\{q(x)+\lambda p(x)\right\}y=0 \quad \ldots\ldots (1)$$

If $y_m(x)$ and $y_n(x)$ are the solutions of for distinct eigen values λ_m and λ_n, then orthogonality is well ensured by the following equation, in the interval [a,b] where the D.E. is real and continuous[9]-[13],

$$\int_a^b p(x)y_m(x)y_n(x)dx=0 \quad \ldots\ldots (2)$$

The following differential equations appear for different parameters p, q and r.

II. STURM-LIOUVILLE EQUATION FOR PARAMETERS AT DIFFERENT DISTINCTION.

A. $\lambda=m^2,\ p=r=1, q=0$

Equation (1) reduces to equation of simple harmonic motion.

$$\frac{d^2y}{dx^2}+m^2y=0 \quad \ldots\ldots(3)$$

B. $\lambda=0,\ p(x)=\frac{1}{(1-x^2)}=\frac{1}{r(x)}, q(x)=n(n+1)$

Equation (1) reduces to Legendre's differential equation

$$\frac{\partial}{\partial x}\left\{(x^2-1)\frac{\partial y}{\partial x}\right\}-n(n+1)y=0 \quad \ldots\ldots(4)$$

C. $\lambda=-m^2, q(x)=n(n+1), r(x)=1-x^2, p(x)=\frac{1}{(1-x^2)}$

Equation (1) reduces to Associated Legendre's differential equation

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial y}{\partial x}\right\}+\left\{n(n+1)-\frac{m^2}{1-x^2}\right\}y=0 \quad \ldots\ldots(5)$$

D. $p = r = x, q = -\dfrac{n^2}{x}, \quad \lambda \to \lambda^2$

Equation (1) yields $x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0$ ……(6)

This is Bessel's differential equation.

E. $p = 1, \lambda = -n(n+1), r = \dfrac{q}{k^2} = x^2$

Equation (1) yields $x^2 y'' + 2xy' + \left[k^2 x^2 - n(n+1)\right] y = 0$ ……(7)

This is Spherical Bessel's differential equation.

F. $p = r = e^{-x^2}, q = 0, \lambda = 2n$

Equation (1) yields $y'' - 2xy' + 2ny = 0$ ……(8)

This is Hermite's differential equation.

G. $p = e^{-x}, q = 0, r = xe^{-x}, \lambda = n$

Equation (1) produces $xy'' + (1-x) y' + ny = 0$ ……(9)

This is Laguerre differential equation.

H. $p = x^m e^{-x}, \lambda = l, q = 0, r = e^{-x} x^{m+1}$

Equation (1) reduces to

$xy'' + (m+1-x) y' + ly = 0$ ……(10)

This is Associated Laguerre differential equation.

All the above equations suggest that we should have some unique differential as well as integral representation for the special functions. In this context Rödrigue relations as differential representations and Schläfli integrals as integral representations are available in literature. We propose unified single formula for such works as integral representations.

III. INTEGRAL REPRESENTATION FOR SPECIAL FUNCTIONS

Motivated by Sturm-Liouville theory and Schläfli integrals, we propose an integral representation for a special function in the following manner:

Let G(x,z) be a continuous differentiable function having Taylor's series expansion, then the special function will appear as coefficients in the following way

$$G(x,z) = \sum_{n=0}^{\infty} z^n g_n(x), \quad \text{…(11)}$$

where *G*(*x,z*) is the generating function of special function $g_n(x)$. The Mellin's transform of complex type of generating function *G*(*x, z*) can be accomplished to produce the special function of $g_n(x)$ i.e.

$$g_n(x) = \frac{1}{2\pi i} \oint dz \; .z^{-n-1} G(x,z) \quad \text{……(12)}$$

Using above axiomatic expression we performed some integrals and the results are given in the following section.

IV. MELLIN'S TRANSFORM OF GENERATING FUNCTION

We choose some generating functions and we obtain that the Equation (12) yields the most expected results.

A. $G(x,z)=\left(1-2xz+z^2\right)^{-\frac{1}{2}}, g_n(x)=P_n(x)$ ……(13)

B. $G(x,z)=e^{\frac{x}{2}(z-\frac{1}{z})}, g_n(x)=J_n(x)$ ……(14)

C. $G(x,z)=\exp\{x^2-(z-x)^2\}, g_n(x)=\dfrac{H_n(x)}{n!}$ ……(15)

D. $G(x,z)=\dfrac{1}{1-z}e^{-\frac{xz}{1-z}}, g_n(x)=\dfrac{L_n(x)}{n!}$ ……(16)

Where $P_n(x)$, $J_n(x)$, $H_n(x)$, and $L_n(x)$ are Legendre, Bessel, Hermite and Laguerre polynomials.

V. CONCLUSION

We found that Mellin's type transform of a well-defined analytic function expressible in Taylor's series yields the special function. The generalized form Equation (11) is sufficient to produce a class of special function for any arbitrary generation function. This axiom may open new route for new class of special function.

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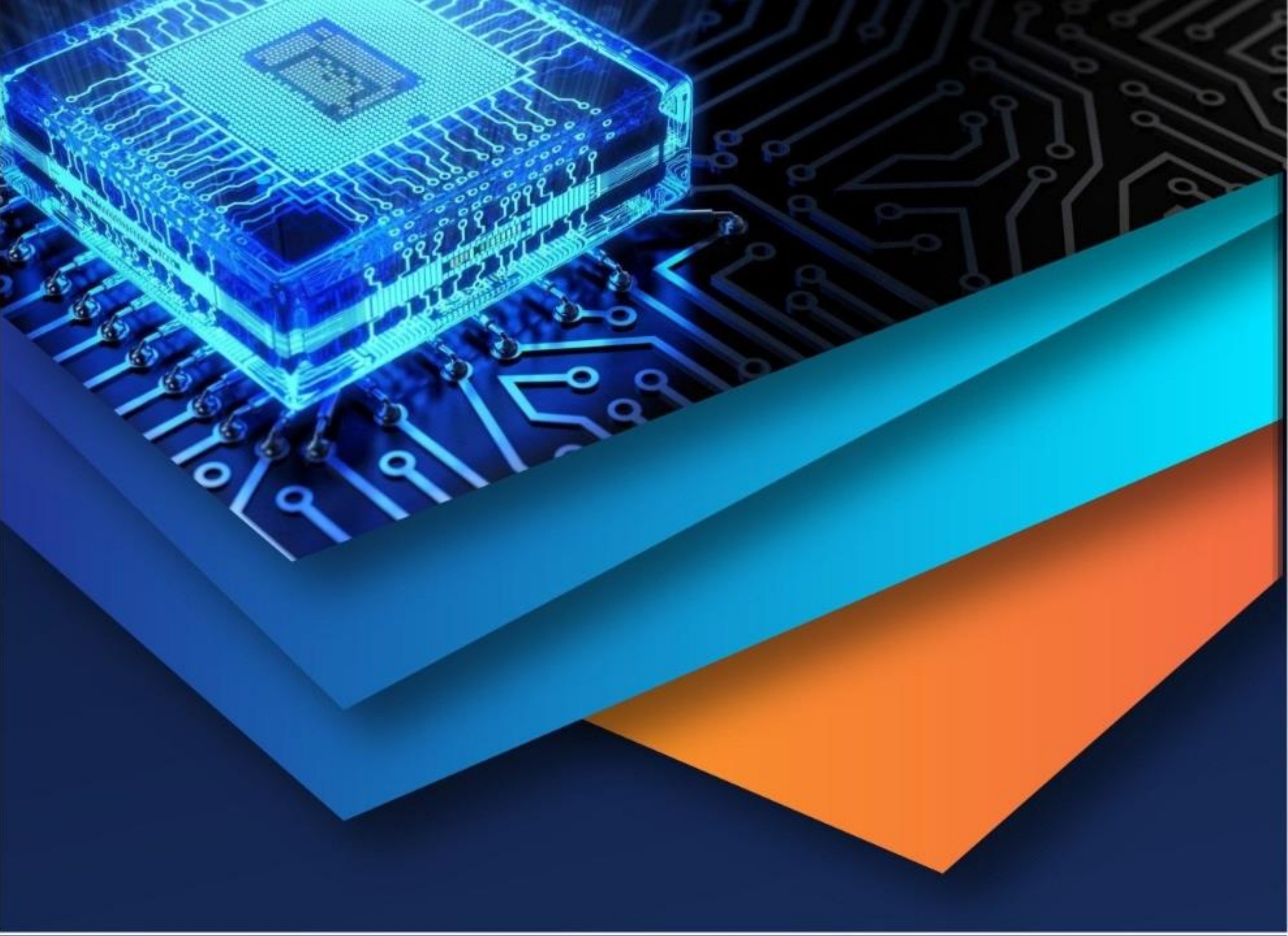

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