iJRASET
International Journal For Research in
Applied Science and Engineering Technology


INTERNATIONAL JOURNAL
FOR RESEARCH
IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

**Volume:** 8 **Issue:** XII **Month of publication:** December 2020

**DOI:** https://doi.org/10.22214/ijraset.2020.32644

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429
Volume 8 Issue XII Dec 2020- Available at www.ijraset.com

# A Note on Residue at Infinity: An Essential Correction to "Limiting Definition"

Ram Swaroop Meghwal[1], Surender Bharadwaj[2]
[1]*Dept. of Physics, Govt. College, Kota (Raj.)- India.*
[2]*Dept. of Maths, Govt. College, Kota (Raj.)- India.*

***Abstract: The basic definitions reviewed. The essential correction to the function is introduced in the definition. Explanations with essential examples are introduced so that the paper exhibits pedagogical and epistemological values. Pseudo natures of analyticity and singularity are considered.***

***Keywords: modified version of Residue at infinity, Pseudo analyticity, . Pseudo singularity***
***PACS numbers: 1.85+f, 2.30-f, 2.30 Fn, 2.90+p***

## I. INTRODUCTION

The theory of complex variables has found everlasting interests for physicists as well as mathematicians.
Vast number of research publications, textbooks and reference books are available. Although an appreciable amount of useful literature already exists, still the research can never be terminating.

## II. REVIEW OF DEFINITIONS AND MODIFICATION:

An analytic function with finite number of isolated singularities in complex plane can be expanded as

$$f(z)=\sum_{r=0}^{n} a_r z^r+\sum_{j=1}^{m}\sum_{s=1}^{k}\frac{b_{js}}{\left(z-z_j\right)^s} \qquad ...(1)$$

Clearly, the function has 'm' poles of order 'k'. In the limit m→∞, the restriction of isolation fails.
If we choose the contour in such a manner that no poles (including pole at infinity too) lie on or within the contour, then according to Cauchy's Integral Theorem, we have [1]

$$\oint_C f(z)dz+\oint_{C_0} f(z)dz+\oint_{C_1} f(z)dz+\oint_{C_2} f(z)dz...=0 \qquad ...(2)$$

Where contour C is positive in counter clockwise direction and all other $C_i$ are negative and on are in clockwise senses. In this way the contours $-C_i$ are positive in counter clockwise directions.

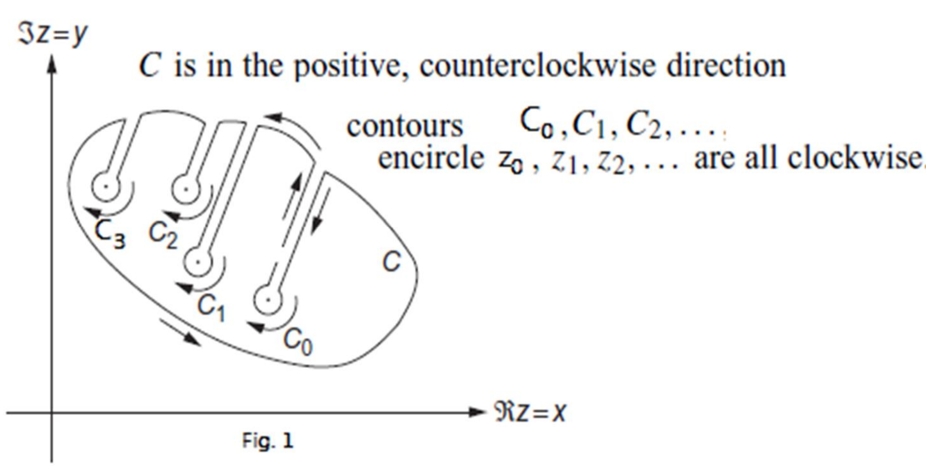


Fig. 1

The integrals around isolated singularities, $z_i \neq \infty$, can be computed using the formula [2]:

$$\oint_{C_i} f(z)dz = -2\pi i b_{i1} \qquad ...(3)$$

When k is finite the residue can be calculated using the formula [3]:

$$\text{Res}\left[f(z), z_i\right] = \lim_{z \to z_i}\left[\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_i)^k f(z)\right\}\right] = b_{i1} \qquad ...(4)$$

When k is not finite, we substitute [4] $z-z_j = t_j$ in Eq. (1)

$$f(z) = \sum_{r=0}^{n} a_r z^r + \sum_{j=1}^{m}\sum_{s=1}^{k=\infty}\frac{b_{js}}{t_j^{\,s}} \qquad ...(5)$$

$$\text{Res}\left[f(z), z_i\right] = b_{i1} = Coeff.of\left(\frac{1}{t_j}\right) \qquad ...(6)$$

Now we concentrate our attention to case where $z_j = \infty$; we redefine Eq. (5)as follows

$$f(z) = f_p + f_P = f_p + f_R + f_N$$

$$f_p = \sum_{r=0}^{n} a_r z^r, \quad f_R = \sum_{j=1}^{m}\frac{b_{j1}}{t_j}, \quad f_N = \sum_{j=1}^{m}\sum_{s=2}^{k=\infty}\frac{b_{js}}{t_j^{\,s}} \qquad ...(7)$$

Where $f_p$ is polynomial part of the function, $f_P$ principal part of the function, , $f_R$ residual part of the function, and , $f_N$ non-residual part of the function. These functions have very distinct features:

$$\lim_{z \to \infty} zf_p \to \infty, \quad \lim_{z \to 0} zf_p \to 0, \qquad a_n = \lim_{z \to 0}\frac{1}{n!}\frac{d^n}{dx^n} f_p$$

$$\lim_{z \to \infty} zf_R \to \sum_{j=1}^{m} b_{j1}, \quad \lim_{z \to 0} zf_R \to b_{j1} = Coeff.of\left(\frac{1}{z-0}\right) = \text{Res}\left[f(z), 0\right]$$

$$\lim_{z \to \infty} zf_N \to 0, \quad \lim_{z \to 0} zf_N \to 0 \qquad ...(8)$$

For poles of order 1, we find from Eq. (4),

$$\text{Res}\left[f(z), z_i\right] = \lim_{z \to z_i}\left\{(z-z_i) f_P(z)\right\} = b_{i1} \qquad ...(9)$$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
*ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429*
*Volume 8 Issue XII Dec 2020- Available at www.ijraset.com*

$$\lim_{z_i \to \infty} f_R(z) = \lim_{z_i \to \infty} \sum \frac{b_{i1}}{z - z_i} \to \lim_{z_i \to \infty} \sum \frac{b_{i1}}{-z_i}$$

$$= -\sum \lim_{z_i \to \infty} \frac{b_{i1}}{z_i} = -\sum_i \lim_{z \to \infty} \frac{b_{i1}}{z} = -\sum_i b_{i1} \lim_{z \to \infty} \frac{1}{z}$$

$$\therefore \lim_{z \to \infty} z f_R(z) = \sum_i b_{i1} \qquad \text{...(10)}$$

Eq.(2) is equivalent to

$$\sum_{\mathrm{i}} \mathrm{Res}\left[f(z), z_i\right] + \mathrm{Res}\left[f(z), \infty\right] = 0$$

$$\Rightarrow \sum_i b_{i1} + \mathrm{Res}\left[f(z), \infty\right] = 0 \qquad \text{...(11)}$$

From Eq. (10) and (11), we see

$$\mathrm{Res}\left[f(z), \infty\right] = -\lim_{z \to \infty} z f_P(z) \qquad \text{...(12)}$$

This is the modified formula for residue at infinity.

$$\boxed{\mathrm{Res}\left[f(z), \infty\right] = -\lim_{z \to \infty} z f\ (z)\ ; \qquad iff \qquad f_p(z) = 0} \qquad \text{...(13)}$$

Thus we observe that Eq. (13) is a particular case of Eq. (12). Moreover, comparing Eq. (9) and (13), it becomes feasible to infer that z= ∞ is regarded as a simple irrespective of any order. This fact is evident from correspondence between *stereographic projections* of *Riemann Sphere* on *Argand Plane*.

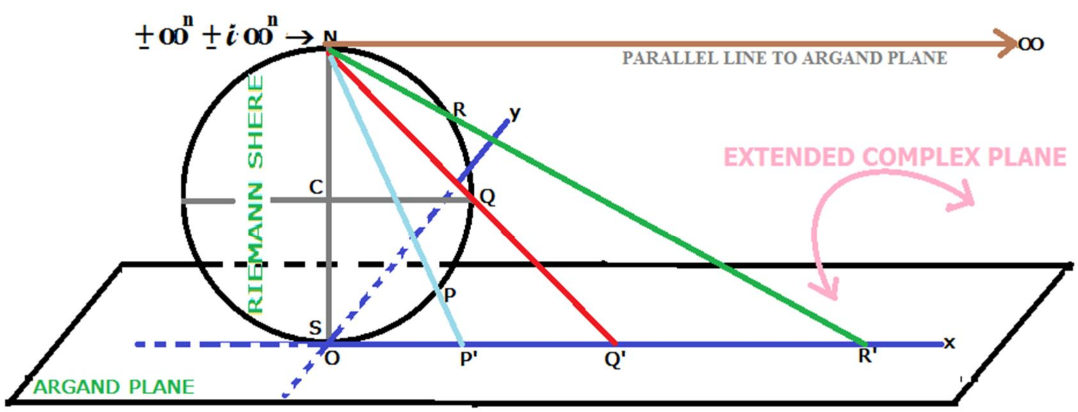


**Fig.2**

Here the origin of Argand Plane ($C^2$) coincides with South Pole of Riemann sphere ($C^3$). The North Pole projects on extended complex plane $\{C^2 \cup \infty\}$. Here it is remarkable that any order of infinity either complex or real corresponds to the unique point (North Pole). Thus we can treat singularity at infinity as a simple pole. And the transformation w=1/z is sufficient. We have no need to utilize $w=1/z^n$.

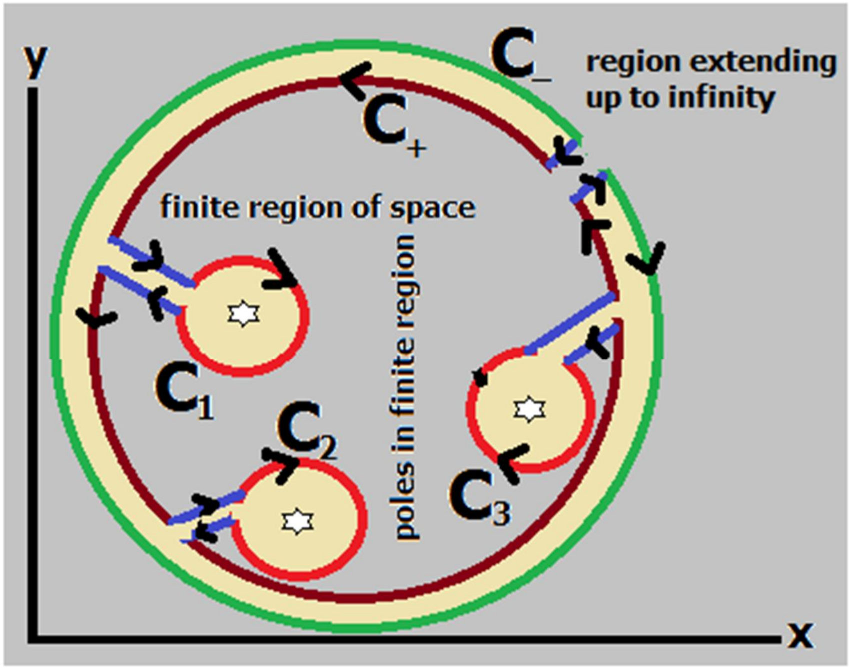


**Fig.3**

Fig.3 shows that poles are encircled by *clockwise* circles. The big inner circle is *counter clockwise* and it is the *counter* considered. The big outer circle is *clockwise* and it should also be utilized to encircle a pole. The poles in the finite region of space are already covered in small circles and therefore this big outermost circle covers the pole at infinity.

$$\text{Res}\left[f(z),\infty\right]=\frac{1}{2\pi i}\int_{C_-} f(z)dz=-\frac{1}{2\pi i}\int_{C_+} f(z)dz \qquad ...(14)$$

## III. IMPOSITION OF MODIFIED CRITERION AND COMPARISON WITH OLD FORMULA:

Here we have table to check the criterion.

| Function | Old criterion | Modified criterion | Remarks |
|---|---|---|---|
| $f(z)=\frac{z}{(z-a)(z-b)}$<br>$\lim_{z\to 0} f(z)=0$<br>$\lim_{z\to\infty} f(z)=0$ | $\lim_{z\to\infty} -zf(z)$<br>$=\lim_{z\to\infty} -z\bullet\frac{z}{(z-a)(z-b)}$<br>$=\lim_{z\to\infty}\frac{-1}{(1-a/z)(z-b/z)}=-1$ | $f(z)=\frac{1}{z}+\frac{a+b}{z^2}+...$<br>$f_p(z)=0,\ f_P(z)=\frac{1}{z}+\frac{a+b}{z^2}+...$<br>$\lim_{z\to\infty} -zf_P(z)=-1$ | Both criteria work well. Function is analytic at infinity but residue is -1. It is *pseudo regularity*. |
| $f(z)=\frac{z^3-z^2+1}{z^3}$<br>$\lim_{z\to 0} f(z)\to\infty$<br>$\lim_{z\to\infty} f(z)=1$ | $\lim_{z\to\infty} -zf(z)$<br>$=-z\bullet\frac{z^3-z^2+1}{z^3}\to\infty$ | $f(z)=1-\frac{1}{z}+\frac{1}{z^3}$<br>$f_p(z)=1,\ f_P(z)=-\frac{1}{z}+\frac{1}{z^3}$<br>$\lim_{z\to\infty} -zf_P(z)=1$ | Hence in old criterion limit doesn't exist while Modified criterion works well. Function has *pseudo analyticity*. |

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
*ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429*
*Volume 8 Issue XII Dec 2020- Available at www.ijraset.com*

| | | | |
|---|---|---|---|
| $f(z)=e^{1/z}$<br>$\lim_{z\to 0} f(z)\to\infty$<br>$\lim_{z\to\infty} f(z)\to 1$ | $\lim_{z\to\infty} -zf(z)\to\infty$ | $f(z)=e^{1/z}$<br>$=1+\frac{1}{z}+\frac{1}{2!z^2}+\dots$<br>$\lim_{z\to\infty} -zf_P(z)=-1$ | Hence in old criterion limit doesn't exist while Modified criterion works well. Function has *pseudo analyticity*. |
| $f(z)=\frac{z^2}{(z-a)(z-b)}$<br>$\lim_{z\to 0} f(z)=0$<br>$\lim_{z\to\infty} f(z)=1$ | $\lim_{z\to\infty} -zf(z)$<br>$=\lim_{z\to\infty} -\frac{z^3}{(z-a)(z-b)}\to\infty$ | $f(z)=1+\frac{a+b}{z}+\dots$<br>$f_p(z)=1,\ f_P(z)=\frac{a+b}{z}+\dots$<br>$\lim_{z\to\infty} -zf_P(z)=-(a+b)$ | Hence in old criterion limit doesn't exist while Modified criterion works well. Function has *pseudo analyticity*. |
| $f(z)=\frac{z^3}{(z-a)(z-b)}$<br>$\lim_{z\to 0} f(z)=0$<br>$\lim_{z\to\infty} f(z)\to\infty$ | $\lim_{z\to\infty} -zf(z)$<br>$=\lim_{z\to\infty} -\frac{z^4}{(z-a)(z-b)}\to\infty$ | $f(z)=z+\{a+b\}+\frac{a^2+ab+b^2}{z}+\dots$<br>$f_P(z)=\frac{a^2+ab+b^2}{z}+\dots$<br>$\lim_{z\to\infty} -zf_P(z)=-(a^2+ab+b^2)$ | Hence in old criterion limit doesn't exist while Modified criterion works well. Function has singularity. |
| $f(z)=\frac{z^3-z+1}{z^3}$<br>$\lim_{z\to 0} f(z)\to\infty$<br>$\lim_{z\to\infty} f(z)=1$ | $\lim_{z\to\infty} -zf(z)$<br>$=-z\bullet\frac{z^3-z+1}{z^3}\to\infty$ | $f(z)=1-\frac{1}{z^2}+\frac{1}{z^3}$<br>$f_p(z)=1,\ f_P(z)=-\frac{1}{z^2}+\frac{1}{z^3}$<br>$\lim_{z\to\infty} -zf_P(z)=0$ | Hence in old criterion limit doesn't exist while Modified criterion works well. Function has *analyticity*. |
| $f(z)=\frac{z^4-z+1}{z^3}$<br>$\lim_{z\to 0} f(z)\to\infty$<br>$\lim_{z\to\infty} f(z)\to\infty$ | $\lim_{z\to\infty} -zf(z)$<br>$=-z\bullet\frac{z^4-z+1}{z^3}\to\infty$ | $f(z)=z-\frac{1}{z^2}+\frac{1}{z^3}$<br>$f_p(z)=z,\ f_P(z)=-\frac{1}{z^2}+\frac{1}{z^3}$<br>$\lim_{z\to\infty} -zf_P(z)=0$ | Hence in old criterion limit doesn't exist while Modified criterion works well. Function has *pseudo singularity*. |

## IV. DISCUSSION AND CONCLUSION:

In old schools of mathematics and physics, calculus of residue is a powerful tool. Singularity at infinite is not given a proper consideration. We find many statements in even in text books [5], [6].[7],[8]that in some cases $z=\infty$ is a pole of order m, though no one applied the transformation $w=1/z^m$, rather $w=1/z$ is utilized. We have given explanation to this point with Fig.2. We observe that it can be regarded as a simple pole equally well. We find very important note as a function may be regular at infinity, but yet have a residue there. The reason is well understood in table provided. Moreover, not only *analyticity and singularity*, but *pseudo analyticity* and *pseudo singularity* also observed and explained. Fig. 3 has its own importance as integral representation of residue at infinity. It makes well understood that *outer most contour* in *anticlockwise* sense encloses the pole at infinity.

ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429
Volume 8 Issue XII Dec 2020- Available at www.ijraset.com

## REFERENCES

[1] Kasper J. Larsen, Robbert Rietkerk , arXiv:1701.01040v4 [hep-th] 26 Sep 2019.
[2] Norbert Hungerbühler1 and Micha Wasem, Hindawi Journal of Mathematics, doi.org/10.1155/2019/6130464.
[3] Ahlfors L. V., Complex analysis. An introduction to the theory of analytic functions of one complex variable, McGraw-Hill Book Co., New York, 1978.
[4] James Ward Brown and Ruel V. Churchill, Complex variables and applications, McGraw-Hill Book Co., New York, 2014.
[5] J. C. Chaturveddi and S. S. Seth, Functions of a Complex Variable, Students' Friends Company, Agra 1986
[6] B. S. Rajput, Mathematical Physics, Pragati Prakashan, Meerut, 2017
[7] B. D. Gupta, Mathematical Physics, Vikash Publishing House, New Delhi, 2009
[8] Satya Prakash, Mathematical Physics, Sultan Chand & Sons, New Delhi, 2014.