iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH
IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 8 Issue: XII Month of publication: December 2020

DOI: https://doi.org/10.22214/ijraset.2020.32650

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429
Volume 8 Issue XII Dec 2020- Available at www.ijraset.com

Commutation and Anti-Commutation Brackets of Quantum Field Theory in Single and Double Cluster Form

Deepmala Meena[1], Ram Swaroop Meghwal[2]
[1, 2]Department of Physics, Government College Kota, Kota, Rajasthan, India

***Abstract:** We present a systematic method to compute commutator of boson and fermion operators. Bosons give simple distributive law with all terms with the same positive sign while fermions show anti distributive laws as alternate signs in distribution and nature of statistics depending on evenness and oddness of total number of fermions.*
***Keywords:** commutations, anti- commutations, slash and backslash notations.*

I. INTRODUCTION

Of course, after tedious efforts nothing is impossible, but to get the solutions in scientific and in sequential way must always be taken as fruitful. With this argument in mind I am starting the work now.

In second quantization, boson operators satisfy Dirac commutation relations and fermion operators satisfy Jordan- Wigner anti - commutation relations. In teaching quantum field theory I realize that we should give students a way to handle the situation in interesting manner.

II. COMMUTATORS

Heisenberg equation of motion [1] states that evolution of the system can be described by time development of a dynamical variable A which has no explicit time dependence in the following manner.

$$i\hbar \frac{dA}{dt} = [A, H]_- \qquad \ldots(1)$$

Where H represents the Hamiltonian of the system. This equation suggest that one should evaluate the commutators. In classical physics dynamical quantities commute and we call them simply c-numbers. Here in quantum world two physical quantities need not be compatible experimentally and corresponding dynamical variables do not commute. When we quantize radiation field we need that conjugate operators associated with canonical coordinate and conjugate momentum should obey the following commutation relations [2].

$$\begin{aligned} &[Q_{k,\alpha}, P_{k',\alpha'}]_- = i\hbar \delta_{k,k'} \delta_{\alpha,\alpha'} \\ &[Q_{k,\alpha}, Q_{k',\alpha'}]_- = 0 = [P_{k,\alpha}, P_{k',\alpha'}]_- \end{aligned} \qquad \ldots(2)$$

Similarly dealing with fermions we observe that Pauli exclusion principle is guaranteed if use the Jordan Wigner anticommutation relations to quantize the Dirac field [3].

$$\begin{aligned} &[b_{k,\alpha}, b^+_{k',\alpha'}]_+ = i\hbar \delta_{k,k'} \delta_{\alpha,\alpha'} \\ &[b_{k,\alpha}, b_{k',\alpha'}]_+ = 0 = [b^+_{k,\alpha}, b^+_{k',\alpha'}]_+ \end{aligned} \qquad \ldots(3)$$

Where bare q- number stands for annihilation operator and with '+' it stands for creation operator. Hamiltonian in general might have a Cluster [4] of destruction and creation operators and we should get expertise in solving commutators of clusters.

III. DIRAC PAULI COMMUTATIONS

We define Dirac Pauli commutation relation as follows

$$[A \quad , \quad B_1]_- = [A \quad , \quad B_1] = AB_1 - B_1A \qquad \ldots(4)$$

To face the cluster I might introduce some unfamiliar notation

$$[A \quad , \quad B_1] = AB_1 - B_1A = \not{B}_1 \qquad \ldots(5)$$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429
Volume 8 Issue XII Dec 2020- Available at www.ijraset.com

Now we should apply this to a simple single cluster

$$\begin{aligned}[A \;,\; B_1B_2] &= AB_1B_2 - B_1B_2A = (AB_1)B_2 - (B_1A)B_2 + B_1(AB_2) - B_1(B_2A) \\ &= [A \;,\; B_1]B_2 + B_1[A \;,\; B_2] = \not{B}_1B_2 + B_1\not{B}_2\end{aligned} \qquad \ldots(6)$$

This ansatz shows that this slash notation is distributive.

$$\begin{aligned}[A \;,\; B_1B_2B_3] &= AB_1B_2B_3 - B_1B_2B_3A = \{(AB_1)B_2B_3 - (B_1A)B_2B_3\} \\ &\quad + \{B_1(AB_2)B_3 - B_1(B_2A)B_3\} + \{B_1B_2(AB_3) - B_1B_2(B_3A)\} \\ &= \not{B}_1B_2B_3 + B_1\not{B}_2B_3 + B_1B_2\not{B}_3\end{aligned} \qquad \ldots(7)$$

Now we can generalize the scheme and we can obtain

$$[A \;,\; B_1B_2\ldots B_n] = AB_1B_2\ldots B_n - B_1B_2\ldots B_nA = \not{B}_1B_2\ldots B_n + B_1\not{B}_2\ldots B_n + B_1B_2\ldots\not{B}_n \qquad \ldots(8)$$

Now we work for a simple double cluster.

$$\begin{aligned}[A_1A_2 \;,\; B_1B_2] &= AB_1B_2 - B_1B_2A = (AB_1)B_2 - (B_1A)B_2 + B_1(AB_2) - B_1(B_2A) \\ &= [A \;,\; B_1]B_2 + B_1[A \;,\; B_2] = [A_1A_2 \;,\; B_1]B_2 + B_1[A_1A_2 \;,\; B_2] \\ &= \left(A_1(A_2B_1) - A_1(B_1A_2) + (A_1B_1)A_2 - (B_1A_1)A_2\right)B_2 \\ &\quad + B_1\left(A_1(A_2B_2) - A_1(B_2A_2) + (A_1B_2)A_2 - (B_2A_1)A_2\right) \\ &= A_1[A_2 \;,\; B_1]B_2 + [A_1 \;,\; B_1]A_2B_2 + B_1A_1[A_2 \;,\; B_2] + B_1[A_1 \;,\; B_2]A_2\end{aligned} \qquad \ldots(9)$$

$$\begin{aligned}[A_1A_2 \;,\; B_1B_2] &= A_1A_2B - BA_1A_2 = \left(A_1(A_2B) - A_1(BA_2)\right) + \left((A_1B)A_2 - (BA_1)A_2\right) \\ &= A_1[A_2 \;,\; B] + [A_1 \;,\; B]A_2 = A_1[A_2 \;,\; B_1B_2] + [A_1 \;,\; B_1B_2]A_2 \\ &= A_1(A_2B_1B_2 - B_1B_2A_2) + (A_1B_1B_2 - B_1B_2A_1)A_2 \\ &= A_1\left((A_2B_1)B_2 - (B_1A_2)B_2 + B_1(A_2B_2) - B_1(B_2A_2)\right) \\ &\quad + \left((A_1B_1)B_2 - (B_1A_1)B_2 + B_1(A_1B_2) - B_1(B_2A_1)\right)A_2 \\ &= A_1[A_2 \;,\; B_1]B_2 + A_1B_1[A_2 \;,\; B_2] + [A_1 \;,\; B_1]B_2A_2 + B_1[A_1 \;,\; B_2]A_2\end{aligned} \qquad \ldots(10)$$

In obtaining Eq. (9) and (10), we have adopted a very easy route as summarized below.

1) First cluster will be paired with second cluster, i. e., no pairing among operators of the same cluster.
2) Expand the bracket and insert element to facilitate pairing and add or subtract the term to cancel it .
3) The process ends up in pairing the last term of the expansion.
4) Use commutators in expansion while pairing may require commutation or anticommutation.

Eq . (9) and (10) can be used to get a symmetric form as follows

$$\begin{aligned}[A_1A_2 \;,\; B_1B_2] &= A_1[A_2 \;,\; B_1]B_2 + \tfrac{1}{2}\{A_1 \;,\; B_1\}[A_2 \;,\; B_2] \\ &\quad + \tfrac{1}{2}\{A_2 \;,\; B_2\}[A_1 \;,\; B_1] + B_1[A_1 \;,\; B_2]A_2\end{aligned} \qquad \ldots(11)$$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429
Volume 8 Issue XII Dec 2020- Available at www.ijraset.com

IV. JORDAN WIGNER ANTI- COMMUTATION RELATIONS

We introduce Jordan Wigner anti- commutation relations in the following way

$$\left[A \;,\; B_1\right]_+ = \left\{A \;,\; B_1\right\} = AB_1 + B_1A \qquad \ldots(12)$$

To face the cluster I might introduce new unfamiliar notation

$$\left\{A \;,\; B_1\right\} = AB_1 + B_1A = \backslash\!\!\!B_1 \qquad \ldots(13)$$

Now we apply the notion of backslash to a simple cluster

$$\begin{aligned}\left[A \;,\; B_1B_2\right] &= AB_1B_2 - B_1B_2A = \left(AB_1\right)B_2 + \left(B_1A\right)B_2 - B_1\left(AB_2\right) - B_1\left(B_2A\right)\\ &= \left\{A \;,\; B_1\right\}B_2 - B_1\left\{A \;,\; B_2\right\} = \backslash\!\!\!B_1B_2 - B_1\backslash\!\!\!B_2\end{aligned} \qquad \ldots(14)$$

Here we observe that backslash introduces alternate sign change in distribution. We should check this point in a big cluster.

$$\begin{aligned}\left[A \;,\; B_1B_2B_3\right] &= AB_1B_2B_3 - B_1B_2B_3A = \left\{\left(AB_1\right)B_2B_3 + \left(B_1A\right)B_2B_3\right\}\\ &\quad - \left\{B_1\left(AB_2\right)B_3 + B_1\left(B_2A\right)B_3\right\} + \left\{B_1B_2\left(AB_3\right) - B_1B_2\left(B_3A\right)\right\}\\ &= \backslash\!\!\!B_1B_2B_3 - B_1\backslash\!\!\!B_2B_3 + B_1B_2/\!\!\!B_3\end{aligned} \qquad \ldots(15)$$

It is a very astonishing situation, with odd number of fermion operators, slash and backslash appear and distributive nature acquires complicacy. Let us check it with even number of fermion operators.

$$\begin{aligned}\left[A \;,\; B_1B_2B_3B_4\right] &= AB_1B_2B_3B_4 - B_1B_2B_3B_4A = \left\{\left(AB_1\right)B_2B_3B_4 + \left(B_1A\right)B_2B_3B_4\right\}\\ &\quad - \left\{B_1\left(AB_2\right)B_3B_4 + B_1\left(B_2A\right)B_3B_4\right\} + \left\{B_1B_2\left(AB_3\right)B_4 + B_1B_2\left(B_3A\right)B_4\right\}\\ &\quad - \left\{B_1B_2B_3\left(AB_4\right) + B_1B_2B_3\left(B_4A\right)\right\}\\ &= \backslash\!\!\!B_1B_2B_3B_4 - B_1\backslash\!\!\!B_2B_3B_4 + B_1B_2\backslash\!\!\!B_3B_4 - B_1B_2B_3\backslash\!\!\!B_4\end{aligned} \qquad \ldots(16)$$

We find that backslash introduces alternate sign change in distribution when cluster contains even number of fermions. We may call it anti- distributive for even number of fermions.

To get further insight into the picture , we do work with cluster of more odd number of operators.

$$\begin{aligned}\left[A \;,\; B_1B_2B_3B_4B_5\right] &= AB_1B_2B_3B_4B_5 - B_1B_2B_3B_4B_5A = \left\{\left(AB_1\right)B_2B_3B_4B_5 + \left(B_1A\right)B_2B_3B_4B_5\right\}\\ &\quad - \left\{B_1\left(AB_2\right)B_3B_4B_5 + B_1\left(B_2A\right)B_3B_4B_5\right\} + \left\{B_1B_2\left(AB_3\right)B_4B_5 + B_1B_2\left(B_3A\right)B_4B_5\right\}\\ &\quad - \left\{B_1B_2B_3\left(AB_4\right)B_5 + B_1B_2B_3\left(B_4A\right)B_5\right\} + B_1B_2B_3B_4\left(AB_5\right) - B_1B_2B_3B_4\left(B_5A\right)\\ &= \backslash\!\!\!B_1B_2B_3B_4B_5 - B_1\backslash\!\!\!B_2B_3B_4B_5 + B_1B_2\backslash\!\!\!B_3B_4B_5 - B_1B_2B_3\backslash\!\!\!B_4B_5 + B_1B_2B_3B_4/\!\!\!B_5\end{aligned} \qquad \ldots(17)$$

Again we observe the same feature as emerged out in Eq. (15).

Now we work for a simple double cluster.

$$\begin{aligned}\left[A_1A_2 \;,\; B_1B_2\right] &= \left[A_1A_2 \;,\; B_1\right]B_2 + B_1\left[A_1A_2 \;,\; B_2\right]\\ &= \left(A_1\left(A_2B_1\right) + A_1\left(B_1A_2\right) - \left(A_1B_1\right)A_2 - \left(B_1A_1\right)A_2\right)B_2\\ &\quad + B_1\left(A_1\left(A_2B_2\right) + A_1\left(B_2A_2\right) - \left(A_1B_2\right)A_2 - \left(B_2A_1\right)A_2\right)\\ &= A_1\left[A_2 \;,\; B_1\right]_+B_2 - \left[A_1 \;,\; B_1\right]A_2B_2 + B_1A_1\left[A_2 \;,\; B_2\right] - B_1\left[A_1 \;,\; B_2\right]A_2\end{aligned} \qquad \ldots(18)$$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429
Volume 8 Issue XII Dec 2020- Available at www.ijraset.com

$$
\begin{aligned}
[A_1A_2 \;,\; B_1B_2] &= A_1[A_2 \;,\; B_1B_2]+[A_1 \;,\; B_1B_2]A_2 \\
&= A_1\big((A_2B_1)B_2+(B_1A_2)B_2-B_1(A_2B_2)-B_1(B_2A_2)\big) \\
&+\big((A_1B_1)B_2+(B_1A_1)B_2-B_1(A_1B_2)-B_1(B_2A_1)\big)A_2 \\
&= A_1[A_2 \;,\; B_1]_+B_2-A_1B_1[A_2 \;,\; B_2]_+ +[A_1 \;,\; B_1]_+B_2A_2-B_1[A_1 \;,\; B_2]_+A_2
\end{aligned}
$$

…(19)

Again Eq . (18) and (19) can be used to get a symmetric form as follows

$$
\begin{aligned}
[A_1A_2 \;,\; B_1B_2] &= A_1[A_2 \;,\; B_1B_2]+[A_1 \;,\; B_1B_2]A_2 \\
&= A_1\big((A_2B_1)B_2+(B_1A_2)B_2-B_1(A_2B_2)-B_1(B_2A_2)\big) \\
&+\big((A_1B_1)B_2+(B_1A_1)B_2-B_1(A_1B_2)-B_1(B_2A_1)\big)A_2 \\
&= A_1[A_2 \;,\; B_1]_+B_2-\tfrac{1}{2}[A_1 \;,\; B_1]_-[A_2 \;,\; B_2]_+-\tfrac{1}{2}[A_2 \;,\; B_2]_-[A_1 \;,\; B_1]_+-B_1[A_1 \;,\; B_2]_+A_2
\end{aligned}
$$

…(20)

V. GENERALIZATION OF COMMUTATION RELATIONS

Now we generalize the above relations so that one handles the situation more easily.

We define the commutator as follows

$$[A \;,\; B]_{-\zeta} = AB-\zeta BA$$

$$\zeta=\begin{cases} + & Boson \\ - & Fermion \end{cases}$$ …(21)

Now imagine commutation as an operation and slash or backslash as an eigen value. We can see

$$\Lambda(\zeta,n)=\begin{cases} / & \forall n & Boson \\ \backslash & n-even & Fermion \\ \backslash\ or\ / & n-odd & \backslash\,\forall\ except\ last\ \&\ /\ for\ last\ fermion \end{cases}$$ …(22)

$$[A \;,\; B_1B_2\dots B_n]=\zeta^0\Lambda B_1B_2\dots B_n+\zeta^1B_1\Lambda B_2\dots B_n+\zeta^{n-1}B_1B_2\dots\Lambda B_n$$ …(23)

Eq. (10c) and (17c) can be used to produce a generalized form as given below.

$$
\begin{aligned}
&[A_1A_2 \;,\; B_1B_2]_- \\
&= A_1[A_2 \;,\; B_1]_{-\zeta}B_2+\tfrac{\zeta}{2}[A_1 \;,\; B_1]_{\zeta}[A_2 \;,\; B_2]_{-\zeta} \\
&\quad +\tfrac{\zeta}{2}[A_2 \;,\; B_2]_{\zeta}[A_1 \;,\; B_1]_{-\zeta}+\zeta B_1[A_1 \;,\; B_2]_{-\zeta}A_2
\end{aligned}
$$

…(24)

VI. APPLICATION IN THE THEORY OF SECOND QUANTIZATION

If ‘s’ is the spin of the particle ,then statistics governing parameter ‘ζ’, can be given as

$$\zeta = sign(-1)^{2s}$$ …(25)

The annihilation and creation are designated as ‘a’ and ‘a^+’. Then Eq. (18) can be used to obtain the following commutators, so that corresponding statistics is obeyed well.

$$[a_i \;,\; a_j^+]_{-\zeta}=\delta_{ij} \;,\; [a_j^+ \;,\; a_i]_{-\zeta}=-\zeta\delta_{ij}\;[a_i \;,\; a_j]_{-\zeta}=0=[a_i^+ \;,\; a_j^+]_{-\zeta}$$

$$a_ia_j^+=\delta_{ij}+\zeta a_j^+a_i \quad , \quad a_ia_j=\zeta a_ja_i \quad ,a_i^+a_j^+=\zeta a_j^+a_i^+$$

With the help of Eq. (18), we can find

$$\left[A \quad , \quad BC\right]_{-} = \left[A \quad , \quad B\right]_{-\zeta} C + \zeta B \left[A \quad , \quad C\right]_{-\zeta} \qquad \ldots(26)$$

I. $\left[a_i \quad , \quad a_j^{+} a_k\right]_{-} = \delta_{ij} a_k$

II. $\left[a_i \quad , \quad a_j^{+} a_k^{+}\right]_{-} = \delta_{ij} a_k^{+} + \zeta \delta_{ik} a_j^{+}$

III. $\left[a_i^{+} \quad , \quad a_j a_k\right]_{-} = -\zeta \delta_{ij} a_k - \delta_{ki} a_j$

IV. $\left[a_i^{+} \quad , \quad a_j^{+} a_k\right]_{-} = -\delta_{ki} a_j^{+}$

V. $\left[a_i \quad , \quad a_j^{+} a_k^{+} a_l a_m\right]_{-} = \left[a_i \quad , \quad a_j^{+} a_k^{+}\right]_{-} a_l a_m = \left(\delta_{ij} a_k^{+} + \zeta \delta_{ik} a_j^{+}\right) a_l a_m$

VI. $\left[a_i^{+} \quad , \quad a_j^{+} a_k^{+} a_l a_m\right]_{-} = a_j^{+} a_k^{+} \left[a_i^{+} \quad , \quad a_l a_m\right]_{-} = -a_j^{+} a_k^{+} \left(\delta_{im} a_l + \zeta \delta_{il} a_m\right)$

…(27)

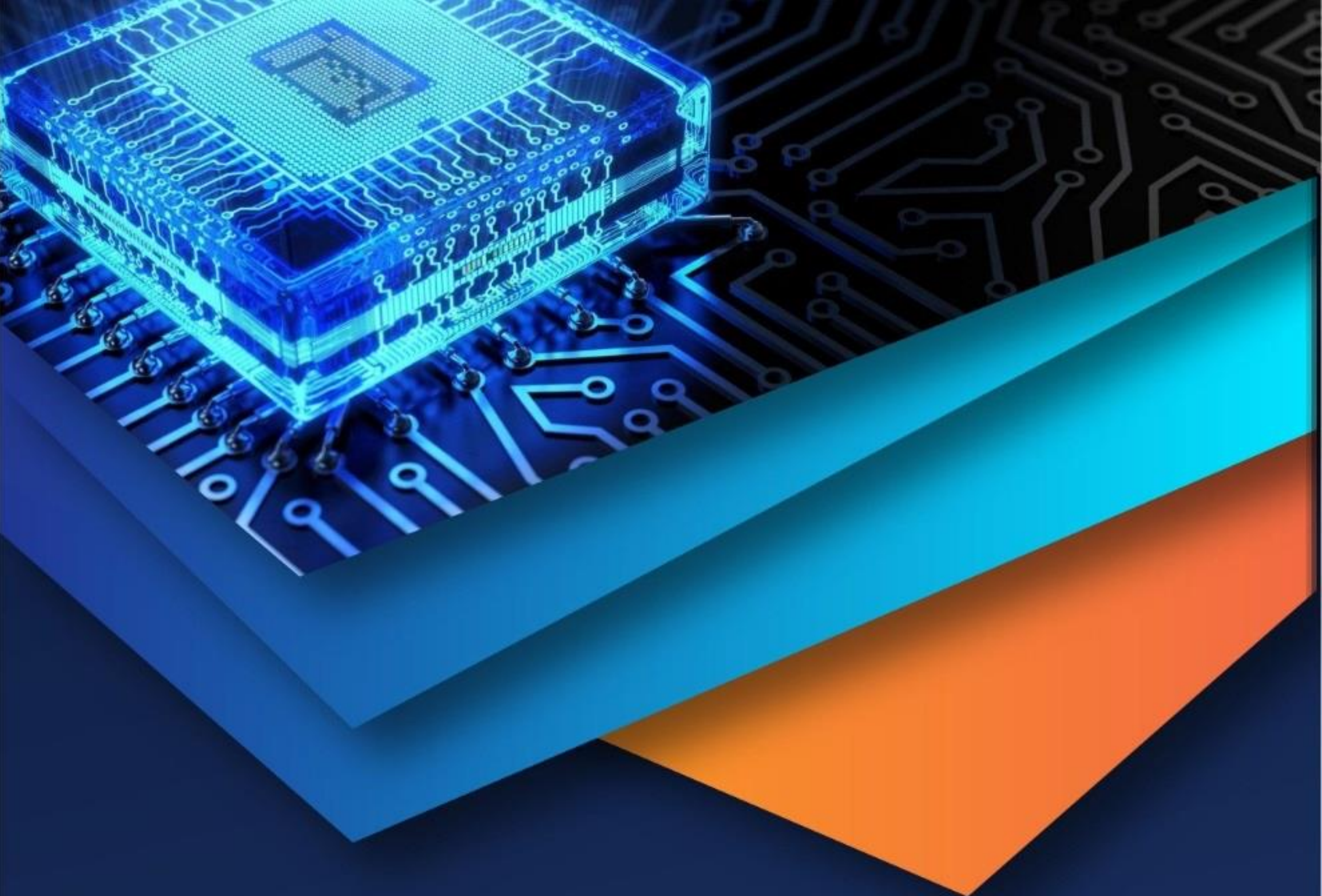
VII. DISCUSSION

These commutation formulas are extremely useful in advanced quantum mechanics , quantum field theory, superconductivity and quantum statistics. The idea of slash and backslash notation is quite beneficial in computing commutator of a cluster.

REFERENCES

[1] Eugene Merzbacher, Quantum Mechanics, John Wiley & Sons NY 1961
[2] W. Heitler, Quantum Theory of Radiation, Clarendon , Oxford 1954
[3] J J Sakurai, Advanced Quantum Mechanics, Addison Wesley, NY 1967
[4] A L Fetter and J D Walecka, Quantum Theory of Many Particle Systems, Mc Graw Hill, NY 1971

* Here the refrences are just to match the results appreared in the paper , the entire methodology is newly developed by the corresponder.



10.22214/IJRASET

45.98

IMPACT FACTOR: 7.129

IMPACT FACTOR: 7.429

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY