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Mittag-Leffler Identity and Matsubara Frequency Sums

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Abstract: Mittag- Leffler identity is obtained from basic definition of meromorphic function. This identity is used to obtain Matsubara frequency sums. Application in the theory of superconductivity is presented as an integral part of finite temperature field theory. The summation is done with the infinite series expansion of coth(πy). We do it from meromorphic function.
Keywords: Mittag- Leffler identity, Green's function technique, Matsubara frequency sums, meromorphic function, finite temperature field theory, infinite series , infinite products

I. INTRODUCTION

It is well known that the powerful mathematical tool for the theoretical investigation of the many – body systems is the Green's function technique [1]. The diagram technique developed for zero temperature cannot be directly used for finite temperatures. The Green's functions needed are temperature Green's functions which are imaginary time dependent. To accomplish the Fourier transform concept of an imaginary frequency comes into play [2]-[3]. The Green's functions have the general property that their values for $\tau < 0$ are related by simple formula to their values for $\tau > 0$. These formulas connote that Green's are nonzero only for [4]

$$\omega_n = \begin{cases} \frac{2n\pi}{\beta} & for\ bosons \\ \frac{(2n+1)\pi}{\beta} & for\ fermions \end{cases} \qquad \ldots(1)$$

; where $\beta = \frac{1}{k_B T}$, k_B = Boltzmann Constant.

Dirac delta function, being even function of argument can be expressed as

$$\delta(\tau) = \frac{1}{\beta} \sum_{n-even} e^{-i\omega_n \tau} \qquad \ldots(2)$$

The Fourier transforms are , in general, defined as

$$f(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} d\omega\, e^{-ib\omega t}\, g(\omega)$$

$$g(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} dt\, e^{ib\omega t}\, f(t) \qquad \ldots(3)$$

Where the Dirac delta is simply (with choice a=1=|b|)

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \qquad \ldots(3)$$

The Matsubara imaginary time and frequency can be introduced as $it \rightarrow \tau$ and $\omega \rightarrow i\,\omega_n$.

Where $\frac{1}{2\pi}\int d\omega .... \rightarrow \frac{1}{\beta}\sum ....$.

Matsubara Green's functions are defined as

$$G_{\sigma,\sigma}(k,\tau) = -\left\langle T_\tau \left\{ c_{k\sigma}(\tau) c^+_{k\sigma}(0) \right\} \right\rangle = -\theta(\tau) \left\langle c_{k\sigma}(\tau) c^+_{k\sigma}(0) \right\rangle = \left\langle \left\langle c_{k\sigma}(\tau); c^+_{k\sigma}(0) \right\rangle \right\rangle$$

$$F_{-\sigma,\sigma}(k,\tau) = -\left\langle T_\tau \left\{ c^+_{-k,-\sigma}(\tau) c^+_{k,\sigma}(0) \right\} \right\rangle = -\theta(\tau) \left\langle c^+_{-k,-\sigma}(\tau) c^+_{k\sigma}(0) \right\rangle \qquad \ldots(4)$$

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The Heaviside- Lorentz step function is introduced as follows

$$\theta(\tau)=\begin{cases}1 & \tau>0\\ 0 & \tau<0\end{cases} \qquad \ldots(5)$$

The time ordering operator is well expressed as

$$T_\tau\{A(\tau)B(\tau')\}=\theta(\tau-\tau')A(\tau)B(\tau')+\zeta\theta(\tau'-\tau)B(\tau')A(\tau) \qquad \ldots(6)$$

Fourier transform of the Green's function $\ll A(\tau);B(\tau')\gg$,which is $\ll A(\tau)|B(\tau')\gg$,satisfies the equation of motion in the following manner

$$i\omega_n\langle\langle A|B\rangle\rangle=\langle[A,B]_{-\zeta}\rangle+\langle\langle[A,H]_-|B\rangle\rangle \qquad \ldots(7)$$

$$[A,B]_{-\zeta}=AB-\zeta BA \qquad \ldots(8)$$

$$\zeta=\begin{cases}+ & for\ bosons\\ - & for\ fermions\end{cases} \qquad \ldots(9)$$

These normal and anomalous Green's functions have the real time Fourier transform as mentioned below

$$\langle\langle A(t);B(t')\rangle\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega(t-t')}\langle\langle A|B\rangle\rangle$$

$$\langle\langle A|B\rangle\rangle=\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle\langle A(t);B(0)\rangle\rangle=G(\omega) \qquad \ldots(10)$$

In the similar fashion the correlation functions are introduced as follows

$$\langle B(t')A(t)\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega(t-t')}\langle BA\rangle$$

$$\langle AB\rangle=\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle A(t)B(0)\rangle \qquad \ldots(11)$$

The correlation functions and Green's functions are related in the following manner [5]

$$F=\langle BA\rangle=\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega\, f_\zeta(\omega)\left[G(\omega+i\in)-G(\omega-i\in)\right],\ \in\to 0+ \qquad \ldots(12)$$

Where $f_\zeta(\omega)=\dfrac{1}{e^{\beta\omega}-\zeta}$ …(13)

The superconducting gap parameter is given by

$$\Delta=\frac{|g|}{N\beta}\sum_{k,n}\langle\langle c^+_{-k\downarrow}|c^+_{k\uparrow}\rangle\rangle=\frac{|g|}{N\beta}\sum_{k,n}\left\{\frac{a_0}{i\omega_n-a}+\frac{b_0}{i\omega_n-b}\right\} \qquad \ldots(14)$$

The density of states is defined as

$$N_\sigma:=\frac{1}{N}\sum_k\frac{\langle\langle c_{k\sigma}|c^+_{k\sigma}\rangle\rangle_{\omega+i\in}-\langle\langle c_{k\sigma}|c^+_{k\sigma}\rangle\rangle_{\omega-i\in}}{2\pi i},\in\to 0+ \qquad \ldots(15)$$

The magnetization parameter can be evaluated by

$$m=\frac{1}{N\beta}\sum_{k,n,\sigma}\sigma G_{\sigma\sigma}(i\omega_n) \qquad \ldots(16)$$

The electronic specific heat is given as

$$C_{es}=\frac{\partial}{\partial T}\frac{1}{N}\sum_k 2(\varepsilon_k-\mu)\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\omega\, f_\zeta(\omega)\left[G(\omega+i\in)-G(\omega-i\in)\right],\ \in\to 0+ \qquad \ldots(17)$$

Now to study the physical properties of the we have to calculate the Green's functions. According to Eq. (14), the problem reduces to compute summation over functions of Matsubara frequencies.

II. MITTAG – LEFFLER IDENTITY [6]

A meromorphic function can be expressed over product of its zeros, and for *sinx* we that $\lim_{x\to o} sinx \cong x$ and $\lim_{x\to 0}\frac{sinx}{x}=1$. Therefore we can expand *sinx* as given below

$$\sin x=\prod_{n=1}^{\infty}(x-0)\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right)=x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right) \qquad \ldots(18)$$

Which with imaginary argument gives

$$\sinh x=x\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2\pi^2}\right) \qquad \ldots(19)$$

Taking natural logarithm and the differentiating w.r.t. x, we get

$$\coth x=\frac{1}{x}+\frac{2}{x}\sum_{n=1}\frac{x^2}{x^2+n^2\pi^2}=\frac{1}{x}+\frac{1}{x}\sum_{n=1}\frac{x^2}{x^2+n^2\pi^2}+\frac{1}{x}\sum_{n=-1}\frac{x^2}{x^2+n^2\pi^2}$$

$$=\frac{1}{x}\left(\frac{x^2}{x^2+n^2\pi^2}\right)_{n=0}+\frac{1}{x}\sum_{n=1}\frac{x^2}{x^2+n^2\pi^2}+\frac{1}{x}\sum_{n=-1}\frac{x^2}{x^2+n^2\pi^2}=\sum_{n=-\infty}^{n=\infty}\frac{x}{x^2+n^2\pi^2} \qquad \ldots(20)$$

This is the required Mittag- Leffler Identity we need for further calculations.

This is the simplest and the most attractive way of getting this identity. In the literature one can find it by so many long and tiring methods. Many research papers might be seen using it [7] , but such interesting derivation is due to this paper only in my opinion.

III. USE OF MITTAG –LEFFLER IDENDITY

Eq. (20) has the Lorentzian structure and we can factorize as follows

$$\sum_{n=-\infty}^{n=\infty}\frac{x}{x^2+n^2}=\frac{1}{2}\sum_{n=-\infty}^{n=\infty}\left(\frac{1}{x-in}+\frac{1}{x+in}\right)=\frac{1}{2}\sum_{n=-\infty}^{n=\infty}\frac{1}{x-in}+\frac{1}{2}\sum_{n=-\infty}^{n=\infty}\frac{1}{x+in}$$

$$=\frac{1}{2}\sum_{n=\infty}^{n=-\infty}\frac{1}{x+in}+\frac{1}{2}\sum_{n=-\infty}^{n=\infty}\frac{1}{x+in}=\sum_{n=-\infty}^{n=\infty}\frac{1}{x+in}$$

$$\therefore \sum_{n=-\infty}^{n=\infty}\frac{1}{x\pm in\pi}=\coth x=\sum_{n=-\infty}^{n=\infty}\frac{\pm 1}{in\pi\pm x} \qquad \ldots(21)$$

Replacing x by y/2 we get the summation for even frequencies

$$\sum_{n=-\infty}^{n=\infty}\frac{1}{i2n\pi+y}=\frac{1}{2}\coth\frac{y}{2}=\sum_{n-even,n=-\infty}^{n=\infty}\frac{(-1)^n}{in\pi+y} \qquad \ldots(22)$$

Replacing y by z+iπ , we obtain the summation for odd frequencies

$$\sum_{n=-\infty}^{n=\infty}\frac{1}{i(2n+1)\pi+z}=\frac{1}{2}\tanh\frac{z}{2}\Leftrightarrow-\frac{1}{2}\tanh\frac{z}{2}=\sum_{n-odd,n=-\infty}^{n=\infty}\frac{(-1)^n}{in\pi+z} \qquad \ldots(23)$$

In this way we are able to calculate sum over Matsubara frequencies.

Some other identities [8] can also be computed as follows

$$\sum_{n=-\infty}^{n=\infty}\frac{1}{x+in\pi}=\coth x=\sum_{n=-\infty}^{n=\infty}\frac{x+in\pi}{x^2+n^2\pi^2}=\sum_{n=-\infty}^{n=\infty}\frac{x}{x^2+n^2\pi^2}+i\pi\sum_{n=-\infty}^{n=\infty}\frac{n}{x^2+n^2\pi^2}$$

Comparing imaginary parts, we have

$$\sum_{n=-\infty}^{n=\infty}\frac{n}{x^2+n^2}=0 \qquad \ldots(24)$$

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Differentiating Eq. (21) w.r.t. x, we get

$$\sum_{n=-\infty}^{n=\infty} \frac{1}{(x \pm in\pi)^2} = \operatorname{csch}^2 x \qquad \text{...(25)}$$

$$\sum_{n=-\infty}^{n=\infty} \frac{1}{y \pm in\pi} \mp \sum_{n=-\infty}^{n=\infty} \frac{1}{x + in\pi} = \coth y \mp \coth x$$

$$\therefore \sum_{n=-\infty}^{n=\infty} \frac{1}{(y \pm in\pi)(x + in\pi)} = \frac{\coth y \mp \coth x}{x \mp y} \qquad \text{...(26)}$$

From Eq.(22) and Eq. (23)

$$\sum_{\forall n, n=-\infty}^{n=\infty} \frac{(-1)^n}{in\pi + x} = \frac{1}{2}\left\{\coth\frac{x}{2} - \tanh\frac{x}{2}\right\} = \csc hx \qquad \text{...(27)}$$

In the same fashion, we can also expand cosx as a meromorphic function and logarithmic derivative with imaginary argument gives

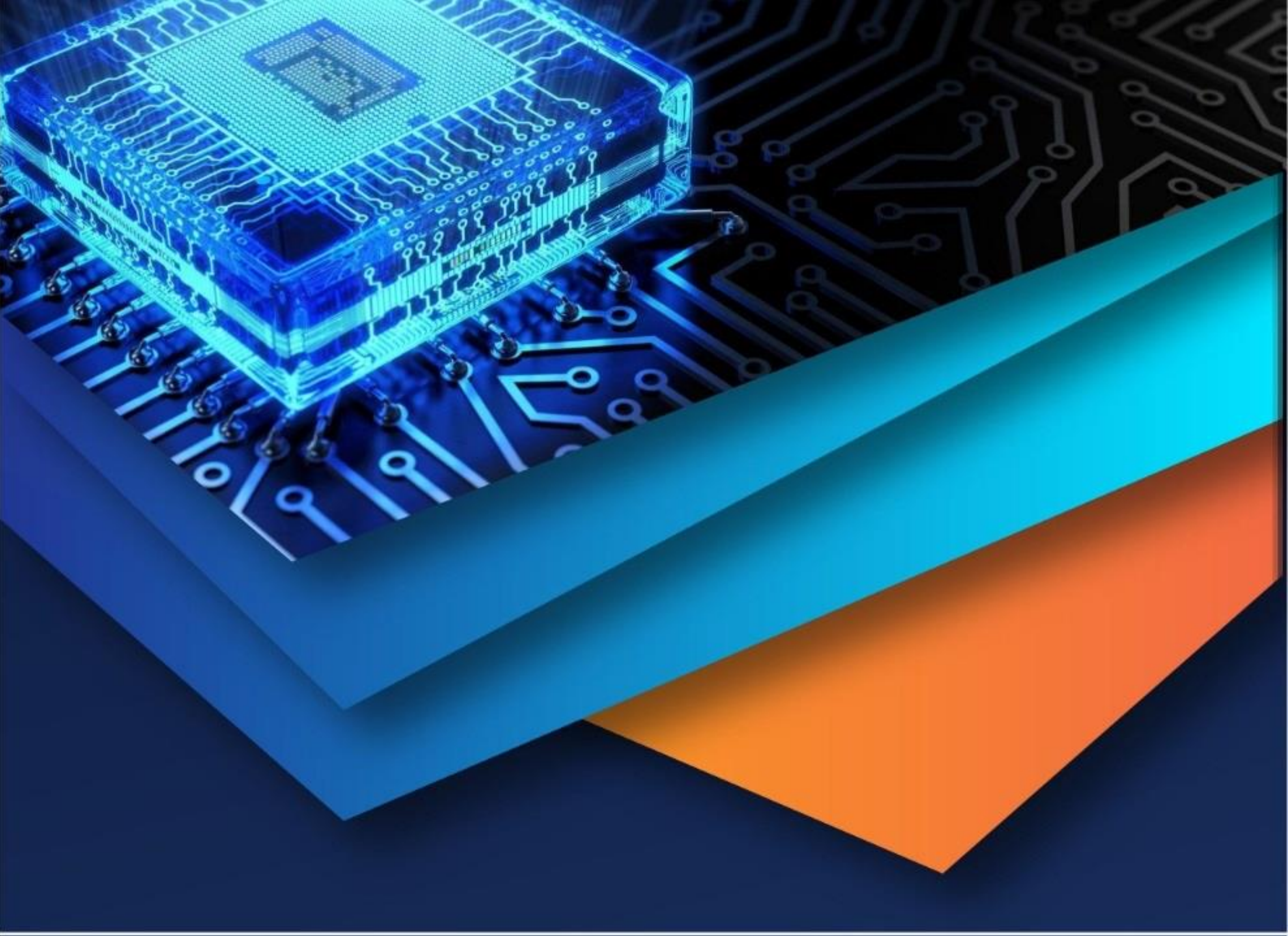
$$\frac{d}{dx}\ln\cosh x = \frac{d}{dx}\ln\left\{\prod_{n=1}^{\infty}\left[1 - \frac{(ix)^2}{\left\{(2n-1)\pi/2\right\}^2}\right]\right\} \Rightarrow \frac{\tanh x}{8x} = \sum_{n=1}^{\infty}\frac{(2n-1)^2\pi^2}{(2n-1)^2\pi^2 + 4x^2} \qquad \text{...(28)}$$

IV. CONCLUSION

The Lorentzian structures of tanhx and cothx are very fruitful identifications. The method of factorization makes the evolving in very simple way. In this way we are capable of performing sum over Matsubara frequencies. The even Matsubara frequencies appear in Bose Einstein statistics obeying ensemble and odd frequency case corresponds to Fermi Dirac statistics.

REFERENCES

[1] A A Abrikosov, L P Gor'kov , I E Dzyaloshiski : Quantum Field Theoretical Methods in Statistical Physics,Pergamon Oxford (1965).
[2] T. Matsubara, Prog. Theor. Phys., 14, 351 (1955)
[3] J. J. Kapusta, Nucl. Phys. B190, 425 (1981);
[4] A L Fetter and J D Walecka, Quantum Theory of Many Particle Systems, Mc Graw – Hill, NY (1971)
[5] R S Meghwal, Ph D Thesis, MDSU, Amer (2008)
[6] I N Sneddon, The Use of integral transforms, , Mc Graw – Hill, NY (1972)
[7] R.Parthasarathy, and Alok Kumar, Physical Review D75, 085007 (2007).
[8] Alok Kumar, Int. J. Pure & App.Math. 113(7), 50-57 (2017)



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