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Gantmacher's Formula

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Abstract: In theory of differential equations we learn to solve (dy/dx)+Py=Q. But Its vector analogue as encountered in case of Lorentz force is extremely tedious and firstly, it was done by Gantmakher. Here a simplified version is presented for scholarly attention.
Keywords: Gantmacher, vector differential equation, Lorentz force problem.

I. INTRODUCTION

General principles of dynamics are either formulated in differential or in integral forms [1]. The differential or integral forms are many times of implicit nature in such a way that solution appears in terms of solution itself [2]. When such form acquire the vector attribute complexity aggravates [3]. Lorentz force [4] is one of such complexities.

II. STATEMENT OF THE PROBLEM AND ITS SOLUTION

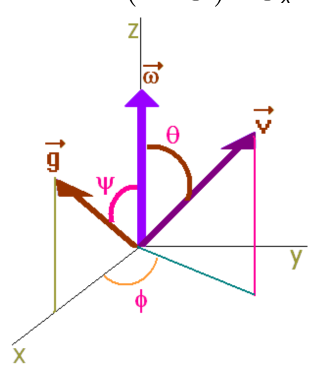
$$m\frac{d^2\vec{r}}{dt^2}=e\left[\vec{E}+\frac{d\vec{r}}{dt}\times\vec{B}\right]\Leftrightarrow\frac{d\vec{v}}{dt}=\frac{e\vec{E}}{m}+\vec{v}\times\frac{e\vec{B}}{m}\equiv\vec{g}+\vec{v}\times\vec{\omega}$$

The cross product can be made to produce two components only if one of the vectors is single component vector. Now velocity vector is not a constant vector, hence we may choose ω to be a single component vector [5]. Moreover, we must allow v to have all three components. There will be no loss of generality if g is confined to x-z plane.

$$\vec{\omega}=\omega\hat{z}=(0,0,\omega)_{\text{rectangular}}=(\omega,0,0)_{spherical}$$
$$\vec{g}=g_x\hat{x}+g_z\hat{z}=(g_x,0,g_z)_{\text{rectangular}}=(g,\psi,0)_{spherical}$$
$$\vec{v}=v_x\hat{x}+v_y\hat{y}+v_z\hat{z}=(v_x,v_y,v_z)_{\text{rectangular}}=(v,\theta,\phi)_{spherical}$$

$$\vec{v}\times\vec{\omega}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\ v_x&v_y&v_z\\ 0&0&\omega\end{vmatrix}=\omega v_y\hat{x}-\omega v_x\hat{y}\Rightarrow\hat{\omega}\times\vec{v}_0=-v_{0y}\hat{x}+v_{0x}\hat{y}$$

$$-\hat{\omega}\times(\hat{\omega}\times\vec{v}_0)=v_{0x}\hat{x}+v_{0y}\hat{y},\ \hat{\omega}\times\vec{g}=g_x\hat{y},\ \ -\hat{\omega}\times(\hat{\omega}\times\vec{g})=g_x\hat{x}$$



$$\frac{d\vec{v}}{dt} = \vec{g} + \vec{v} \times \vec{\omega} \Rightarrow \frac{dv_x}{dt} = g_x + \omega v_y \qquad ...(1)$$

$$\frac{dv_y}{dt} = -\omega v_x \qquad ...(2)$$

$$\frac{dv_z}{dt} = g_z \qquad ...(3)$$

Differentiating Eq. (1) and using Eq. (2) , we obtain

$$\frac{d^2v_x}{dt^2} = \frac{dg_x}{dt} + \omega\frac{dv_y}{dt} = 0 + \omega(-\omega v_x) \Rightarrow \frac{d^2v_x}{dt^2} + \omega^2 v_x = 0$$

$$\therefore v_x = A\cos\omega t + B\sin\omega t\,, \quad v_{0x} = A\cos 0^o + B\sin 0^o = A$$

$$v_x = v_{0x}\cos\omega t + B\sin\omega t$$

$$\frac{dv_x}{dt} = g_x + \omega v_y \Rightarrow v_y = -\frac{g_x}{\omega} + \frac{1}{\omega}\frac{d}{dt}\left\{v_{0x}\cos\omega t + B\sin\omega t\right\}$$

$$\therefore v_y = -\frac{g_x}{\omega} - v_{0x}\sin\omega t + B\cos\omega t\,, \quad v_{0y} = -\frac{g_x}{\omega} + B$$

$$\boxed{v_x = v_{0x}\cos\omega t + v_{0y}\sin\omega t + \frac{g_x}{\omega}\sin\omega t} \qquad ---(4)$$

$$v_y = -\frac{g_x}{\omega} - v_{0x}\sin\omega t + \left(v_{0y} + \frac{g_x}{\omega}\right)\cos\omega t$$

$$\boxed{v_y = -v_{0x}\sin\omega t + v_{0y}\cos\omega t + \frac{g_x}{\omega}\left(\cos\omega t - 1\right)} \qquad ---(5)$$

$$\boxed{v_z = v_{0z} + g_z t} \qquad ---(6)$$

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Further results will be obtained after integrating Eqs. (4)-(6)

$$\frac{dx}{dt}=v_{0x}\cos\omega t+v_{0y}\sin\omega t+\frac{g_x}{\omega}\sin\omega t$$

$$\Rightarrow x=v_{0x}.\frac{\sin\omega t}{\omega}+v_{0y}.\frac{-\cos\omega t}{\omega}+\frac{g_x}{\omega}.\frac{-\cos\omega t}{\omega}+c$$

$$\Rightarrow x_0=v_{0y}.\frac{-1}{\omega}+\frac{g_x}{\omega}.\frac{-1}{\omega}+c\Rightarrow c=x_0+\frac{v_{0y}}{\omega}+\frac{g_x}{\omega^2},$$

$$\therefore x=x_0+v_{0x}.\frac{\sin\omega t}{\omega}+v_{0y}.\frac{1-\cos\omega t}{\omega}+\frac{g_x}{\omega}.\frac{1-\cos\omega t}{\omega}$$

$$\Rightarrow x=x_0+v_{0x}.\frac{\sin\omega t-\omega t+\omega t}{\omega}+v_{0y}.\frac{1-\cos\omega t}{\omega}+\frac{g_x}{\omega}.\frac{1-\cos\omega t-\frac{1}{2}(\omega t)^2+\frac{1}{2}(\omega t)^2}{\omega}$$

$$\Rightarrow \boxed{x=x_0+v_{0x}t+\frac{1}{2}g_xt^2+v_{0x}\left(\frac{\sin\omega t-\omega t}{\omega}\right)-v_{0y}\left(\frac{\cos\omega t-1}{\omega}\right)-g_x\left(\frac{\cos\omega t-1+\frac{1}{2}(\omega t)^2}{\omega^2}\right)}$$

$$\frac{dy}{dt}=-v_{0x}\sin\omega t+v_{0y}\cos\omega t+\frac{g_x}{\omega}(\cos\omega t-1)$$

$$\Rightarrow y=v_{0x}\frac{\cos\omega t}{\omega}+v_{0y}\frac{\sin\omega t}{\omega}+\frac{g_x}{\omega}\left(\frac{\sin\omega t}{\omega}-t\right)+D$$

$$\Rightarrow y_0=\frac{v_{0x}}{\omega}+D\Rightarrow D=y_0-\frac{v_{0x}}{\omega},$$

$$\therefore y=y_0+v_{0x}\frac{\cos\omega t-1}{\omega}+v_{0y}\frac{\sin\omega t-\omega t+\omega t}{\omega}+g_x\left(\frac{\sin\omega t-\omega t}{\omega^2}\right)$$

$$\Rightarrow y=y_0+v_{0y}t+v_{0x}\frac{\cos\omega t-1}{\omega}+v_{0y}\frac{\sin\omega t-\omega t}{\omega}+g_x\left(\frac{\sin\omega t-\omega t}{\omega^2}\right)$$

$$\Rightarrow \boxed{y=y_0+v_{0y}t+v_{0x}\left(\frac{\cos\omega t-1}{\omega}\right)+\left\{v_{0y}+\frac{g_x}{\omega}\right\}\left(\frac{\sin\omega t-\omega t}{\omega}\right)}$$

$$\frac{dz}{dt}=v_{0z}+g_zt\Rightarrow \boxed{z=z_0+v_{0z}t+g_z\frac{t^2}{2}}$$

$$\vec{r} = x_0\hat{x} + v_{0x}\hat{x}t + \frac{1}{2}g_x\hat{x}t^2 + v_{0x}\hat{x}\left(\frac{\sin\omega t - \omega t}{\omega}\right) - v_{0y}\hat{x}\left(\frac{\cos\omega t - 1}{\omega}\right)$$

$$-g_x\hat{x}\left(\frac{\cos\omega t - 1 + \frac{1}{2}(\omega t)^2}{\omega^2}\right) + y_0\hat{y} + v_{0y}\hat{y}t + v_{0x}\hat{y}\left(\frac{\cos\omega t - 1}{\omega}\right)$$

$$+\left\{v_{0y}\hat{y} + \frac{g_x}{\omega}\hat{y}\right\}\left(\frac{\sin\omega t - \omega t}{\omega}\right) + z_0\hat{z} + v_{0z}\hat{z}t + g_z\hat{z}\frac{t^2}{2}$$

$$= \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2 + \left(\frac{\cos\omega t - 1}{\omega}\right)\left\{-v_{0y}\hat{x} + v_{0x}\hat{y}\right\}$$

$$+\left(\frac{\cos\omega t - 1 + \frac{1}{2}(\omega t)^2}{\omega^2}\right)\left\{-g_x\hat{x}\right\} + \left(\frac{\sin\omega t - \omega t}{\omega}\right)\left\{v_{0x}\hat{x} + v_{0y}\hat{y} + \frac{g_x}{\omega}\hat{y}\right\}$$

$$= \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2 + \left(\frac{\cos\omega t - 1}{\omega}\right)\left\{\hat{\omega} \times \vec{v}_0\right\}$$

$$+\left(\frac{\cos\omega t - 1 + \frac{1}{2}(\omega t)^2}{\omega^2}\right)\left\{\hat{\omega} \times (\hat{\omega} \times \vec{g})\right\}$$

$$+\left(\frac{\sin\omega t - \omega t}{\omega}\right)\left\{-\hat{\omega} \times (\hat{\omega} \times \vec{v}_0) + \frac{\hat{\omega} \times \vec{g}}{\omega}\right\}$$

$$= \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2 + \left(\frac{\cos\omega t - 1}{\omega^2}\right)\left\{\vec{\omega} \times \vec{v}_0\right\}$$

$$+\left(\frac{\cos\omega t - 1 + \frac{1}{2}(\omega t)^2}{\omega^4}\right)\left\{\vec{\omega} \times (\vec{\omega} \times \vec{g})\right\}$$

$$+\left(\frac{\sin\omega t - \omega t}{\omega^3}\right)\left\{\vec{\omega} \times \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{v}_0)\right\}$$

$$\frac{d\vec{v}}{dt} = \vec{g} - \vec{\omega} \times \frac{d\vec{r}}{dt} \Rightarrow \vec{v} = \vec{g}t - \vec{\omega} \times \vec{r} + \vec{c} \Rightarrow \vec{v}_0 = -\vec{\omega} \times \vec{r}_0 + \vec{c}$$

$$\Rightarrow \boxed{\vec{v} = \vec{v}_0 + \vec{g}t - \vec{\omega} \times (\vec{r} - \vec{r}_0)}$$

III. DISCUSSION

The problem involving vector nature of differential equation is dealt by reducing it to a planar structure. Then solution is obtained in somewhat easier way. Again combining the solutions of two components of planar domain, we get the general solution. This is the simplest way of getting Gantmacher's formula.

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