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On Fixed Point Result in Double Controlled Metric Spaces

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Abstract: In this paper, we introduce Hardy and Rogers type contractions in the class of double controlled metric spaces and establish fixed point theorem. Our result are generalization of some known results of literature. We also provide example to illustrate significance of the established result .
Keywords: fixed point, a double controlled type metric, controlled type metric, extended b- metric space.

I. INTRODUCTION

The notion of a b- metric spaces was studied by Bakhtin [1], Czerwik [2] and many fixed point results were obtained for single and multivalued mappings by Czerwik and many other auothers. (see 3, 4, 5 6, 7, 8) The generalizations of b- metric spaces Kamran et al. [9] and others (see 10, 11, 12, 13) was introduced extended b- metric spaces by controlling the triangle inequality rather than using control function in the contractive conditions. Proving extentions of Banach contraction principle from metric spaces to b- metric spaces and hence to controlled metric type spaces. Recent article [11] introduced double controlled metric type spaces and prove Banach Contraction Principle and Kannan[14] type contraction in double controlled metric type spaces which is generalized the results of [12], [13].

In this paper we first define Hardy and Rogers type contractions [17] in the setting of double controlled metric type spaces and prove fixed point results. We also provide example to illustrate significance of the established results.

II. PRELIMINARIES

Definition 2.1 [9] Given a functions $\theta : X \times X \to [1, \infty)$, where X is a nonempty set. The function $p : X \times X \to [0, \infty)$ is called an extended b- metric if

1. $p(x, y) = 0$ if and only if $x = y$,
2. $p(x, y) = p(y, x)$,
3. $p(x, y) \le \theta(x, y)\,[p(x, z) + p(z, y)]$, for all x, y, z ε X.

Mlaiki et al. [11] generalized the notion of b – metric spaces.

Definition 2.2 [11] Given a functions $\alpha : X \times X \to [1, \infty)$, where X is a nonempty set. Let function $q : X \times X \to [0, \infty)$. Suppose that

1. $q(x, y) = 0$ if and only if $x = y$,
2. $q(x, y) = q(y, x)$,
3. $q(x, y) \le \alpha(x, z)\, q(x, z) + \alpha(z, y)\, q(z, y)$, for all x, y, z ε X.
   Then q is called a controlled metric type and (X, q) is called a controlled metric type spaces.

Now , we introduce a more general b- metric space.

Definition 2.3 [12] Given non-comparable functions $\alpha, \beta : X \times X \to [1, \infty)$. The function $d : X \times X \to [0, \infty)$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \le \alpha(x, z)\, d(x, z) + \beta(z, y)\, d(z, y)]$, for all x, y, z ε X.

Then d is called a double controlled metric type by α and β.

Remark 2.1 A controlled metric type is also a double controlled metric type when taking the same function. The converse is not true in general**.**

Example 2.1 [12] Let $X = \{ 0, 1, 2 \}$. Consider the double controlled type metric $d : X \times X \to [0, \infty)$ defined by

$d(0, 0) = d(1, 1) = d(2, 2) = 0$, $d(0, 1) = d(1, 0) = 1$, $d(0, 2) = d(2, 0) = 1/2$, $d(1, 2) = d(2, 1) = 2/5$

and $\alpha, \beta : X \times X \to [1, \infty)$ defined by

$\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = \alpha(0, 2) = \alpha(2, 0) = 1$, $\alpha(0, 1) = \alpha(1, 0) = 11/10$, $\alpha(1, 2) = \alpha(2, 1) = 8/5$.

$B(0, 0) = \beta(1, 1) = \beta(2, 2) = 1$, $\beta(0, 1) = \beta(1, 0) = 11/10$, $\beta(0, 2) = \beta(2, 0) = 3/2$, $\beta(1, 2) = \beta(2, 1) = 5/4$.

Note that,

$d(0, 1) > \alpha(0, 2)\, d(0, 2) + \alpha(2, 1)\, d(2, 1)$.

Thus d is not a controlled metric type for the function α.

Definition 2.4 [12] Let (X, d) be a double controlled metric type spaces by one or two functions

1. The sequence $\{ x_n \}$ is convergent to some $x \in X$, if for each positive ε, there is some integer N such that $d(x_n, x) < \varepsilon$ for each $n \geq N$. It is also written as $\lim_{n\to\infty} x_n = x$.
2. The sequence $\{ x_n \}$ is said Cauchy if for every $\varepsilon > 0$, $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$, where N is some integer .
3. (X, d) is said complete if every Cauchy sequence is convergent.

Definition 2.5 [12]] Let (X, d) be a double controlled metric type spaces by either one function or two functions for $x \in X$ and $k > 0$.

1. We define $B(x, k)$ as
$$B(x, k) = \{ y \in X , d(x, y) < k . \}$$
2. The self map T on X is said to be continuous at x in X if for all $\delta > 0$, there exists $k > 0$ such that
$$T(B(x, k)) \subset B(Tx, \delta).$$
Note that if T is continuous at x in (X, d), then $x_n \to x$ implies that $Tx_n \to Tx$, when $n \to \infty$.

III. MAIN RESULT

Result on Hardy and Rogers type contractions.

Theorem 3.1 Let (X, d) be a complete double controlled metric type spaces with $\alpha,\beta : X\times X \to [1,\infty)$. If $T : X \to X$ satisfies the inequality

$$d(Tx, Ty) \leq a\, d(x, y) + b\, d(x, Tx) + c\, d(y, Ty) + e\, d(x,Ty) + f\, d(y,Tx), \quad \ldots\ldots \quad 1$$

where $a, b, c, e, f \geq 0$ and for each $x_0 \in X$,

$a + b + c + (e+f)/2 \lim_{n, m\to\infty} \alpha(x_n,x_m) + (e+f)/2 \lim_{n, m\to\infty} \beta(x_n, x_m) < 1$, and $\sup_{m \geq 1} \beta(x_{i+1}, x_m)\, \alpha(x_{i+1}, x_{i+2}) / \alpha(x_i, x_{i+1}) < 1 / k$. Then T has a unique fixed point.

Proof – The considered sequence $\{ x_n \}$ verifies $x_{n+1} = Tx_n$ for all $n \in N$. Obviously, if there exists $n_0 \in N$ for which $x_{n+1} = x_n$, then $Tx_n = x_n$ and the proof is finished. Thus, we suppose that $x_{n+1} \neq x_n$ for every $n \in N$. Thus by (1), we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$$
$$\leq a\, d(x_n, x_{n-1}) + b\, d(x_{n-}, x_{n+1}) + c\, d(x_{n-1}, x_n) + e\, d(x_n, x_n) + f\, d(x_{n-1-}, x_{n+1})$$
$$= a\, d(x_n, x_{n-1}) + b\, d(x_{n-}, x_{n+1}) + c\, d(x_{n-1}, x_n) + f\, d(x_{n-1-}, x_{n+1})$$

$\leq$ a d (x_n, x_{n-1}) + b d (x_{n-}, x_{n+1}) + c d (x_{n-1}, x_n) + f α (x_{n-1}, x_n) d (x_{n-1-}, x_n) +
f β (x_n, x_{n+1}) d (x_n, x_{n+1})

d (x_{n+1}, x_n) $\leq$ [a +c + f α (x_{n-1}, x_n)] d (x_{n-1}, x_n) + [b+ f β (x_n, x_{n+1})] d (x_n, x_{n+1}). …… 2

Similarly,

d (x_n, x_{n+1}) = d (Tx_{n-1}, Tx_n)

$\leq$ a d (x_{n-1}, x_n) + b d (x_{n-1-}, x_n) + c d (x_n, x_{n-1}) +e d (x_{n-1}, x_{n+1}) +f d (x_{n-}, x_n)

$\leq$ a d (x_{n-1}, x_n) + b d (x_{n-1-}, x_n) + c d (x_n, x_{n+1}) + e α (x_{n-1}, x_n) d (x_{n-1}, x_n) +eβ (x_n, x_{n+1}) d (x_n, x_{n+1})

d (x_n, x_{n+1}) $\leq$ [a +b + e α (x_{n-1}, x_n)] d (x_{n-1}, x_n) + [c+ e β (x_n, x_{n+1})]d (x_n, x_{n+1}). …… 3

Adding (2) and (3_, We get

2 d (x_n, x_{n+1}) $\leq$ [2a +b +c + (e + f)α (x_{n-1}, x_n)] d (x_{n-1}, x_n) + [b + c + (e + f) β (x_n, x_{n+1})]d (x_n, x_{n+1})

[2 – b – c - (e+f) β (x_n, x_{n+1})] d (x_n, x_{n+1}) $\leq$ [2a +b +c+(e + f)α (x_{n-1}, x_n)] d (x_{n-1}, x_n)

$$d(x_n, x_{n+1}) \leq \frac{2a+b+c+(e+f)\alpha(x_{n-1},x_n)]}{[2-b-c-(e+f)\beta(n_n,x_{n+1})]} d(x_{n-1}, x_n)$$

$\leq$ k d (x_{n-1}, x_n) ………. 4

Where $k = \frac{2a+b+c+(e+f)\alpha(x_{n-1},x_n)]}{[2-b-c-(e+f)\beta(n_n,x_{n+1})]} < 1.$

a + b + c + (e+f)/2 $\lim_{n, m\to\infty}$ α (x_n, x_m) + (e+f)/2 $\lim_{n, m\to\infty}$ β (x_n, x_m) < 1

2a + 2b + 2c + (e+f) $\lim_{n, m\to\infty}$ α (x_n, x_m) + (e+f) $\lim_{n, m\to\infty}$ β (x_n, x_m) < 2

2a + b + c + (e+f) $\lim_{n, m\to\infty}$ α (x_n, x_m) $\leq$ 2- b- c - (e+f) $\lim_{n, m\to\infty}$ β (x_n, x_m)

Implies, $\frac{2a+b+c+(e+f)\alpha(x_{n-1},x_n)]}{[2-b-c-(e+f)\beta(n_n,x_{n+1})]} < 1.$

Implies, k < 1.

Thus, we have

d (x_n, x_{n+1}) $\leq$ k^n d (x_0, x_1) . ……… 5

For all n,m ε N (n< m), we have

d (x_n, x_m) $\leq$ α (x_n, x_{n+1}) d (x_n, x_{n+1}) + β (x_{n+1}, x_m) d (x_{n+1}, x_m)

$\leq$ α (x_n, x_{n+1}) d (x_n, x_{n+1}) + β (x_{n+1}, x_m) α (x_{n+1}, x_{n+2}) d (x_{n+1}, x_{n+2}) + β (x_{n+1}, x_m) β (x_{n+2}, x_m) d (x_{n+2}, x_m)

$\leq$ α (x_n, x_{n+1}) d (x_n, x_{n+1}) + β (x_{n+1}, x_m) α (x_{n+1}, x_{n+2}) d (x_{n+1}, x_{n+2}) + β (x_{n+1}, x_m) β (x_{n+2}, x_m) α (x_{n+2}, x_{n+3}) d (x_{n+2}, x_{n+3})

+ β (x_{n+1}, x_m) β (x_{n+2}, x_m) β (x_{n+3}, x_m) d (x_{n+3}, x_m)

$\leq$ …

$\leq$ α (x_n, x_{n+1}) d (x_n, x_{n+1}) $+\sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\beta(x_j,x_m))\,\alpha(x_i,x_{i+1})d(x_i,x_{i+1}) + \prod_{k=n+1}^{m-1}\beta(x_k,x_m)\,d(x_{m-1},x_m)$……… 6

$\leq$ α (x_n, x_{n+1}) k^n d (x_0, x_1) $+\sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\beta(x_j,x_m))\,\alpha(x_i,x_{i+1})\,k^i\,d(x_0,x_1) + \prod_{k=n+1}^{m-1}\beta(x_k,x_m)k^{m-1}d(x_0,x_1)$ ……….. 7

$\leq$ α (x_n, x_{n+1}) k^n d (x_0, x_1) $+\sum_{i=n+1}^{m-1}(\prod_{j=0}^{i}\beta(x_j,x_m))\,\alpha(x_i,x_{i+1})\,k^i\,d(x_0,x_1)$.

Let $S_l = \sum_{i=0}^{l}(\prod_{j=0}^{i}\beta(x_j,x_m))\,\alpha(x_i,x_{i+1})\,k^i\,d(x_0,x_1)$………… 8

Consider

$V_i = \prod_{j=0}^{i} \beta(x_j, x_m)\alpha(x_i, x_{i+1})\, k^i\, d(x_0, x_1)$. ………… 9

We have

$V_{i+1} / V_i = \beta(x_{i+1}, x_m)\ \alpha(x_{i+1}, x_{i+2})\, k\ / \alpha(x_i, x_{i+1})$. ……… 10

In view of condition (2) and the ratio test, we ensure that the series $\sum v_i$ converges. Thus, $\lim_{n\to\infty} S_n$ exists. Hence, the real sequence$\{S_n\}$ is Cauchy.

Now, using (6), we get

$d(x_n, x_m) \le d(x_0, x_1)[k^n \alpha(x_n, x_{n+1}) + (S_{m-1} - S_n)]$. …… 11

Above, we used $\alpha(x, y) \ge 1$. Letting $n, m \to\infty$ in (11) we obtain

$$\operatorname{Lim}_{n, m\to\infty} d(x_n, x_m) = 0. \quad \ldots\ldots\ldots\ldots \qquad 12$$

Thus , the sequence $\{x_n\}$ is Cauchy in the complete double controlled metric space (X, d). So, there is some $x^* \varepsilon X$, so that

$$\operatorname{Lim}_{n, m\to\infty} d(x_n, x^*) = 0. \quad \ldots\ldots \qquad 13$$

that is , $x_n \to x^*$ as $n\to\infty$. Now, we will prove that x^* is a fixed point of T. By (1) we get

$$\begin{aligned} d(x^*, Tx^*) &\le \alpha(x^*, x_{n+1})\, d(x^*, x_{n+1}) + \beta(x_{n+1}, Tx^*)\, d(x_{n+1}, Tx^*) \\ &= \alpha(x^*, x_{n+1})\, d(x^*, x_{n+1}) + \beta(x_{n+1}, Tx^*)\, d(Tx_n, Tx^*) \\ &\le \alpha(x^*, x_{n+1})\, d(x^*, x_{n+1}) + \beta(x_{n+1}, Tx^*)[a\, d(x_n, x^*) + b\, d(x_n, x_{n+1}) + c\, d(x^*, Tx^*) + e\, d(x_n, Tx^*) + f\, d(x^*, Tx_n)]. \end{aligned}$$

Taking the limit as $n\to\infty$ and using (13), and the fact that $\lim_{n\to\infty} \alpha(x_n, x)$ and $\lim_{n\to\infty} \beta(x, x_n)$ exist are finite , we obtain that $d(x^*, Tx^*) \le [(c + e + f) \lim_{n\to\infty} \beta(x_{n+1}, Tx^*)]\, d(x^*, Tx^*)$]. …… 14

Suppose that $x^* \ne Tx^*$ having in mind that $(c + e + f) \lim_{n\to\infty} \beta(x_{n+1}, Tx^*) < 1$.

So, $0 < d(x^*, Tx^*) \le [(c + e + f) \lim_{n\to\infty} \beta(x_{n+1}, Tx^*)]\, d(x^*, Tx^*) < d(x^*, Tx^*)$.

It is contradiction. This yields that $x^* = Tx^*$.

Let x^{**} in X bee such that $Tx^{**} = x^{**}$ and $x^* \ne x^{**}$. We have

$$\begin{aligned} 0 < d(x^*, x^{**}) &= d(Tx^*, Tx^{**}) \\ &\le a\, d(x^*, x^{**}) + b\, d(x^*, Tx^*) + c\, d((x^{**}, Tx^{**}) + e\, d(x^*, Tx^{**}) + f\, d(x^{**}, Tx^*) \\ &\le (a + e + f)\, d(x^*, x^{**}). \end{aligned}$$

It is a contradiction, so $x^* = x^{**}$. Hence x^* is the unique fixed point of T.

Remark 3.1 The assumption (1) in Theorem 1 above can be replace by assumption that the mapping T and the double controlled metric d are continuous. Indeed, when $x_n \to x^*$, $Tx_n \to Tx^*$ and hence we have
$\operatorname{Lim}_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(Tx_{n+1}, Tx^*) = d(x^*, Tx^*)$ and hence

$$Tx^* = x^*.$$

Theorem 3.1 is illustrated by the following example

Example 3.1 We endow $X = \{0, 1, 2\}$ by the following double controlled metric type space
$d : X \times X \to [0, \infty)$ defined by
$d(0,0) = d(1,1) = d(2,2) = 0$, $d(0,1) = d(1,0) = 1$, $d(0,2) = d(2,0) = 2/5$, $d(1,2) = d(2,1) = 6/25$
and $\alpha, \beta : X \times X \to [1, \infty)$ defined by
$\alpha(0,0) = \alpha(1,1) = \alpha(2,2) = 1$, $\alpha(0,2) = \alpha(2,0) = 151/100$, $\alpha(0,1) = \alpha(1,0) = 6/5$,
$\alpha(1,2) = \alpha(2,1) = 8/5$.
$\beta(0,0) = \beta(1,1) = \beta(2,2) = 1$, $\beta(0,1) = \beta(1,0) = 6/5$, $\beta(0,2) = \beta(2,0) = 8/5$,
$\beta(1,2) = \beta(2,1) = 33/20$.
Then d is double controlled metric type space but d is not a controlled metric type space for the function α.
Indeed,
$d(0,1) = 1 > 247/250 = \alpha(0,2)\, d(0,2) + \alpha(2,1)\, d(2,1)$.
Choose function $T : X \to X$ such that
$T0 = 2$ and $T1 = T2 = 1$, set
$a = 3/25$, $b = 2/25$, $c = 1/25$, $e = 4/25$ and $f = 5/25$.
It is clear that condition (1) is satisfied. In addition (2)and (3) holds for each x_0 in X. All hypothesis of theorem (3.1) are fulfilled. Here $x^* = 1$ is the unique fixed point.

Theorem 3.2 Let (X, d) be a double controlled metric type spaces with $\alpha, \beta : X \times X \to [1,\infty)$. If $T : X \to X$ satisfies the inequality

$$d(Tx, Ty) \leq a\, d(x, y) + b\, d(x, Tx) + c\, d(y, Ty) + e\,[d(x,Ty) + d(y,Tx)], \quad \ldots\ldots \quad 14$$

where a, b, c, e, ≥ 0 and for each $x_0 \,\varepsilon\, X$,
$a + b + c + e \lim_{n\to\infty} \alpha(x_n, x_m) + e \lim_{n\to\infty} \beta(x_n, x_m) < 1$, and
$\sup_{m \geq 1} \beta(x_{i+1}, x_m)\, \alpha(x_{i+1}, x_{i+2}) \,/\, \alpha(x_i, x_{i+1}) < 1/k$.
Then T has a unique fixed point.

Remark 3.2
1. In case $b = c = e = f = 0$, we get a result due to Theorem 1 of [12].
2. In case $a = 0$, $b = c$, $e = f = 0$, we get a result due to Theorem 3 of [12].
3. In case $\alpha(x,y) = \beta(x, y)$, $d = e = 0$, we get a result due to Theorem 8 of [13].
4. In case $\alpha(x,y) = \beta(x, y)$, $b = c = d = e = 0$, we get a result due to Theorem 1 of [11].
5. In case $\alpha(x,y) = \beta(x, y)$, $a = 0$, $b = c$, $d = e = 0$, we get a result due to Theorem 2 of [11].

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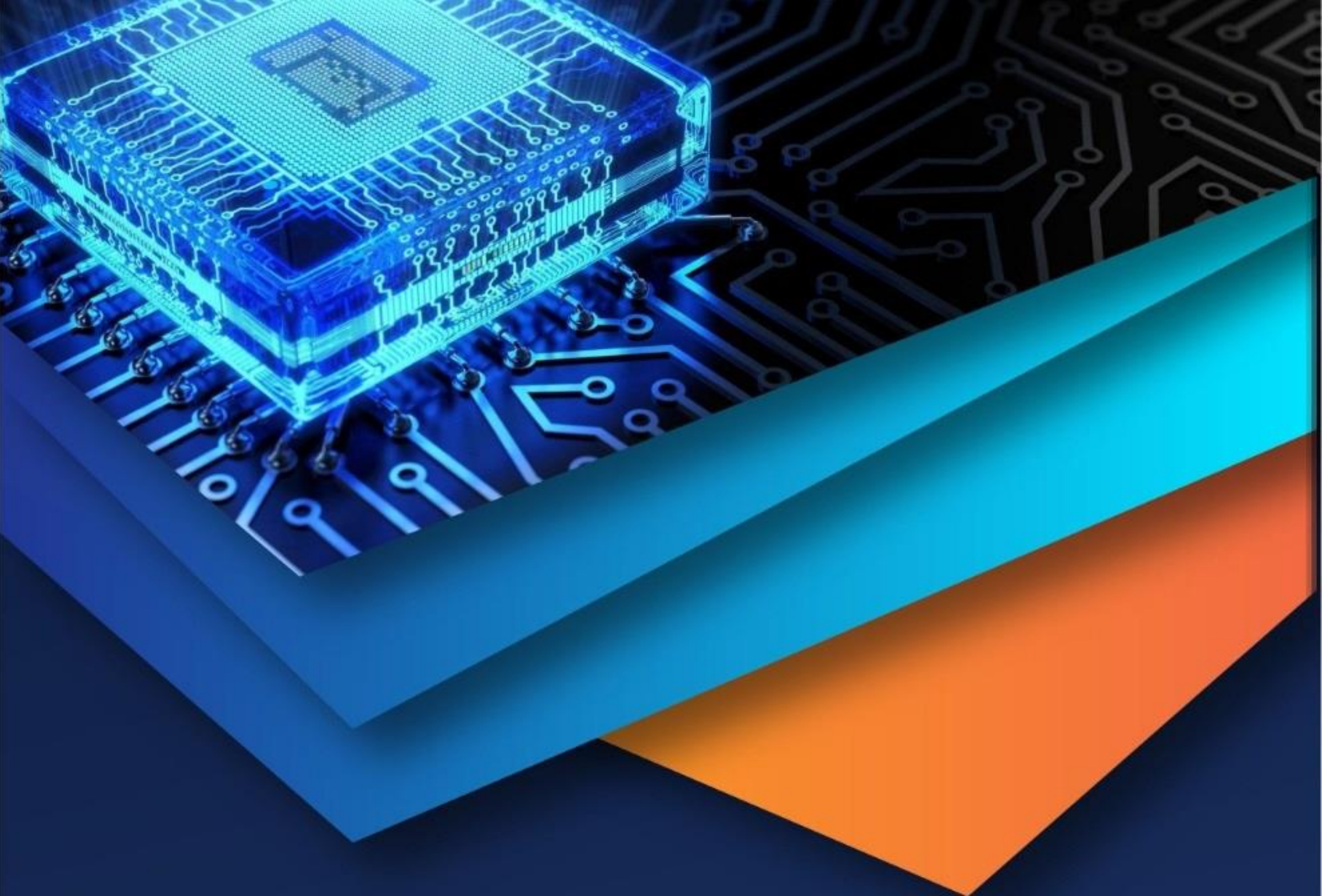

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