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Iterative Methods for the Solution of Semi-Nonlinear Systems with Linear Diagonals

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Abstract: We discuss Jacobi, Gauss-seidel and SOR methods for the solution of semi-nonlinear systems with linear diagonals in this paper. A Comparison of these methods is done through an example.
Keywords: Non-linear equations, Iterative methods, Jacobi , Gauss-seidel, SOR.

I. INTRODUCTION

Let us consider a system of n non-linear equations in n unknownsof the form

$$\left.\begin{aligned} &f_1(x_1, x_2, \ldots\ldots, x_n) = 0 \\ &f_2(x_1, x_2, \ldots\ldots, x_n) = 0 \\ &. \\ &. \\ &f_n(x_1, x_2, \ldots\ldots, x_n) = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

If one can express the system(1.1) as in the following from

$$\left.\begin{aligned} &a_{11} f_{11}(x_1) + a_{12} f_{12}(x_2) + \ldots\ldots\ldots\ldots + a_{1n} f_{1n}(x_n) = b_1 \\ &a_{21} f_{21}(x_1) + a_{22} f_{22}(x_2) + \ldots\ldots\ldots\ldots + a_{2n} f_{2n}(x_n) = b_2 \\ &. \\ &. \\ &a_{n1} f_{n1}(x_1) + a_{n2} f_{n2}(x_2) + \ldots\ldots\ldots\ldots + a_{nn} f_{nn}(x_n) = b_n \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

then, the system(1.2) can be called as semi-nonlinear system.

In the system (1.2), the functions $f_{11}(x_1), f_{22}(x_2), \ldots\ldots\ldots\ldots, f_{nn}(x_n)$ are linear in $x_1, x_2, \ldots\ldots, x_n$,then the system (1.2) can be called as semi non-linear system with linear diagonals.

We now write the semi non-linear system with linear diagonals as

$$\left.\begin{aligned} &a_{11} x_1 + a_{12} f_{12}(x_2) + \ldots\ldots\ldots\ldots + a_{1n} f_{1n}(x_n) = b_1 \\ &a_{21} f_{21}(x_1) + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2n} f_{2n}(x_n) = b_2 \\ &. \\ &. \\ &a_{n1} f_{n1}(x_1) + a_{n2} f_{n2}(x_2) + \ldots\ldots\ldots\ldots + a_{nn} x_n = b_n \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.3)$$

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We assume throughout this paper that the matrix obtained from (1.3) i.e.,

$$A=\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix}. \qquad \text{............(1.4)}$$

is a positive definite matrix.

II. ITERATIVE METHODS

we now discuss the Jacobi, Gauss-seidel and SOR methods for solving the semi-non linear system with linear diagonals i.e., the system (1.3).

A. Jacobi Method

Firstly, we re-write system (1.3) as

$$\left.\begin{array}{l} x_1+\dfrac{a_{12}}{a_{11}} f_{12}(\mathrm{x}_2)+.............+\dfrac{a_{1n}}{a_{11}} f_{1n}(\mathrm{x}_n)=\dfrac{b_1}{a_{11}} \\ \dfrac{a_{21}}{a_{22}} f_{21}(\mathrm{x}_1)+x_2.............+\dfrac{a_{2n}}{a_{22}} f_{2n}(\mathrm{x}_n)=\dfrac{b_2}{a_{22}} \\ . \\ . \\ \dfrac{a_{n1}}{a_{nn}} f_{n1}(\mathrm{x}_1)+\dfrac{a_{n2}}{a_{nn}} f_{n2}(\mathrm{x}_2).............+x_n=\dfrac{b_n}{a_{nn}} \end{array}\right\} \text{....................(2.1)}$$

Forming a matrix A_s by collecting the coefficients of the variables as well as functions, we have

$$A_s=\begin{bmatrix} 1 & \dfrac{a_{12}}{a_{11}} & . & . & . & \dfrac{a_{12}}{a_{11}} \\ \dfrac{a_{21}}{a_{22}} & 1 & . & . & . & \dfrac{a_{2n}}{a_{22}} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \dfrac{a_{n1}}{a_{nn}} & \dfrac{a_{n2}}{a_{nn}} & . & . & . & 1 \end{bmatrix} \qquad \text{.............. (2.2)}$$

Splitting the matrix A_s as

$$A_s = I - L_s - U_s \text{ .... } (2.3)$$

where $-L_s$ *and* $-U_s$ *are strictly lower and upper triangular parts of the matrix* A_s *respectevily*. Now, the Jacobi matrix for the semi non-linear system (2.1) is

$$J_s = (L_s + U_s) \text{ .... } (2.4)$$

Let λ_i *be the eigen values of the jacobi matrix* J_s *such that*

$$-1 < \lambda_i < 1 \text{ .... } (2.5)$$

Let the maximum eigen values of the matrix J_s in magnitude i.e., the spectral radius of J_s be $\overline{\mu_s}$. Then ,we have

$$\rho(\mathrm{J}_s) = \max_{(\mathrm{i}=1,2,.....,\mathrm{n})} \left|\lambda_i(\mathrm{J}_s)\right| = \overline{\mu_s} \text{ .... } (2.6)$$

The Jacobi method for the solution of the system (2.1) is given by

$$\left.\begin{aligned}
x_1^{(k+1)} &= \left(b_1 - a_{12} f_{12}(x_2^k) - \text{............} - a_{1n} f_{1n}(x_n^k)\right) / a_{11} \\
x_2^{(k+1)} &= \left(b_2 - a_{21} f_{21}(x_1^k) - \text{............} - a_{2n} f_{2n}(x_n^k)\right) / a_{22} \\
&. \\
x_n^{(k+1)} &= \left(b_n - a_{n1} f_{n1}(x_1^k) - \text{............} - a_{n,n-1} f_{n,n-1}(x_{n-1}^k)\right) / a_{nn}
\end{aligned}\right\} \text{...................} (2.7)$$

$$(\mathrm{k} = 0,1,2,....)$$

This method (2.7) *converges as long as* $\overline{\mu_s}$ *of* (2.6) is less than one.

B. Gauss-Seidel Method

The Gauss-Seidel method for the system (2.1) is given by

$$\left.\begin{aligned}
x_1^{(k+1)} &= \left(b_1 - a_{12} f_{12}(x_2^k) - a_{13} f_{13}(x_3^k) \text{............} - a_{1n} f_{1n}(x_n^k)\right) / a_{11} \\
x_2^{(k+1)} &= \left(b_2 - a_{21} f_{21}(x_1^{k+1}) - a_{23} f_{23}(x_3^k) \text{............} - a_{2n} f_{2n}(x_n^k)\right) / a_{22} \\
&. \\
&. \\
x_n^{(k+1)} &= \left(b_n - a_{n1} f_{n1}(x_1^{k+1}) - a_{n2} f_{n2}(x_2^{k+1}) \text{............} - a_{n,n-1} f_{n,n-1}(x_{n-1}^{k+1})\right) / a_{nn}
\end{aligned}\right\} \text{..................} (2.8)$$

$$(k = 0,1,2,........)$$

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The Gauss-Seidel iterative matrix is

$$G_s = (\mathrm{I} - \mathrm{L}_s)^{-1} U_s \text{ .... } (2.9)$$

where L_s and U_s are as defined in (2.3).
This method converges as long as the spectral radius of G_s in magnitude is less than one i.e.,

$$\rho(G_s) < 1 \text{ .... } (2.10)$$

C. Successive Over Relaxation (SOR) Method

The SOR method for the solution of (2.1) is given by

$$\left.\begin{aligned}
x_1^{(k+1)} &= (1-\omega)x_1^{(k)} - \omega\frac{a_{12}}{a_{11}} f_{12} x_2^{(k)} - \text{..........} - \omega\frac{a_{1n}}{a_{11}} f_{1n} x_n^k + \omega b_1 \\
x_2^{(k+1)} &= -\omega\frac{a_{21}}{a_{22}} f_{21} x_1^{(k+1)} + (1-\omega)x_2^{(k)} - \text{..........} - \omega\frac{a_{2n}}{a_{22}} f_{2n} x_n^k + \omega b_2 \\
&. \\
&. \\
x_n^{(k+1)} &= -\omega\frac{a_{n1}}{a_{nn}} f_{n1} x_1^{(k+1)} - \omega\frac{a_{n2}}{a_{nn}} f_{n2} x_2^{(k+1)} - \text{..........} - \omega\frac{a_{n,n-1}}{a_{nn}} f_{n,n-1} x_{n-1}^{k+1} + (1-\omega)x_n^k + \omega b_n
\end{aligned}\right\} \text{.....(2.11)}$$

$$(\mathrm{k} = 0, 1, 2, \text{..........})$$

where, the choice for the relaxation parameter ω of SOR method is

$$\omega = \frac{2}{1+\sqrt{1-(\overline{\mu_s})^2}} \text{.......................} (2.12)$$

where $\overline{\mu_s}$ *is as defined in* (2.6).

TheSORmethod (2.11) in matrix notation is

$$\begin{bmatrix}
1 & 0 & . & . & . & 0 \\
\omega\frac{a_{21}}{a_{22}}\frac{f_{21}(x_1)}{x_1} & 1 & . & . & . & 0 \\
\omega\frac{a_{31}}{a_{33}}\frac{f_{31}(x_1)}{x_1} & \omega\frac{a_{32}}{a_{33}}\frac{f_{32}(x_2)}{x_2} & 1 & . & . & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
\omega\frac{a_{n1}}{a_{nn}}\frac{f_{n1}(x_1)}{x_1} & \omega\frac{a_{n2}}{a_{nn}}\frac{f_{n2}(x_2)}{x_2} & . & . & . & 1
\end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ x_n \end{pmatrix}^{(K+1)} =$$

$$\begin{bmatrix} 1-\omega & -\omega\frac{a_{12}}{a_{11}}\frac{f_{12}(x_2)}{x_2} & . & . & . & -\omega\frac{a_{1n}}{a_{11}}\frac{f_{1n}(x_n)}{x_n} \\ 0 & 1-\omega & . & . & . & -\omega\frac{a_{2n}}{a_{22}}\frac{f_{2n}(x_n)}{x_n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1-\omega \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}^{(k)} + \omega \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

The SOR iterative matrix is

$$S_s = (\mathrm{I} - \omega \mathrm{L}_s)^{-1}\{(1-\omega)\mathrm{I} + \omega \mathrm{U}_s\} \text{ .... } (2.13)$$

This method converges if

$$\rho(\mathrm{S}_s) < 1 \text{ .... } (2.14)$$

III. NUMERICAL EXAMPLES

A. *Example 3.1*

We consider a semi non-linear system with linear diagonals i.e.,

$$\left.\begin{array}{l} 20x_1 - x_2^3 - x_3^2 = 18 \\ -x_1^3 + 7x_2 - 2x_3 = 4 \\ -x_1^2 - 2x_2^2 + 10x_3 = 7 \end{array}\right\} \text{..........} (3.1)$$

whose exact solution is a unit vector.

The matrix A_s *for the system* (3.1) as *obtained in* (2.2) *i.e.,*

$$A_s = \begin{bmatrix} 20 & -1 & -1 \\ -1 & 7 & -2 \\ -1 & -2 & 10 \end{bmatrix} \text{ .... } (3.2)$$

is positive definite and the eigen values of Jacobi matrix J_s i.e.,

$$\mathrm{J}_s = \begin{bmatrix} 0 & 1/20 & 1/20 \\ 1/7 & 0 & 2/7 \\ 1/10 & 2/10 & 0 \end{bmatrix} \text{ .... } (3.3)$$

are 0.281822,0.042332 and 0.239490 and hence $\overline{\mu_s} = 0.281822$. The relaxation parameter ω_s of SOR method as defined in (2.12), is obtained as

$$\omega_s = 1.02068588 \text{ ....} (3.4)$$

The methods discussed in this paper are applied to obtain the solution of (3.1) up to an error less than 0.5×10^{-9} taking a null vector as an initial guess and

the results obtained are tabulated below along with the error $E = \sqrt{\sum_{i=1}^{n} |1 - x_i|}$.

Table-1 Iterative compressions

Methods	No. Of iterations took for the convergence (n)	Error (E)
Jacobi	30	$0.33675133e^{-4}$
Gauss-Seidel	17	$0.1083196e^{-4}$
SOR	15	$0.15692699e^{-4}$

B. Example 3.2

For the following semi nonlinear system with linear diagonals

$$\left.\begin{array}{l} 20x_1 - x_2^3 - x_3^2 = 18 \\ -x_1^2 - 2x_2 + 10x_3 = 7 \\ -x_1^3 + 7x_2 - 2x_3 = 4 \end{array}\right\} \ldots\ldots\ldots\ldots(3.2)$$

the *coffiecient* matrix A_s *is*

$$A_s = \begin{bmatrix} 20 & -1 & -1 \\ -1 & -2 & 10 \\ -1 & 7 & -2 \end{bmatrix} \ldots\ldots\ldots\ldots(3.3)$$

and the jacobi matrix J_s *is*

$$J_s = \begin{bmatrix} 0 & 0.05 & 0.05 \\ 0.5 & 0 & -5 \\ 0.5 & -3.5 & 0 \end{bmatrix} \ldots\ldots\ldots(3.4)$$

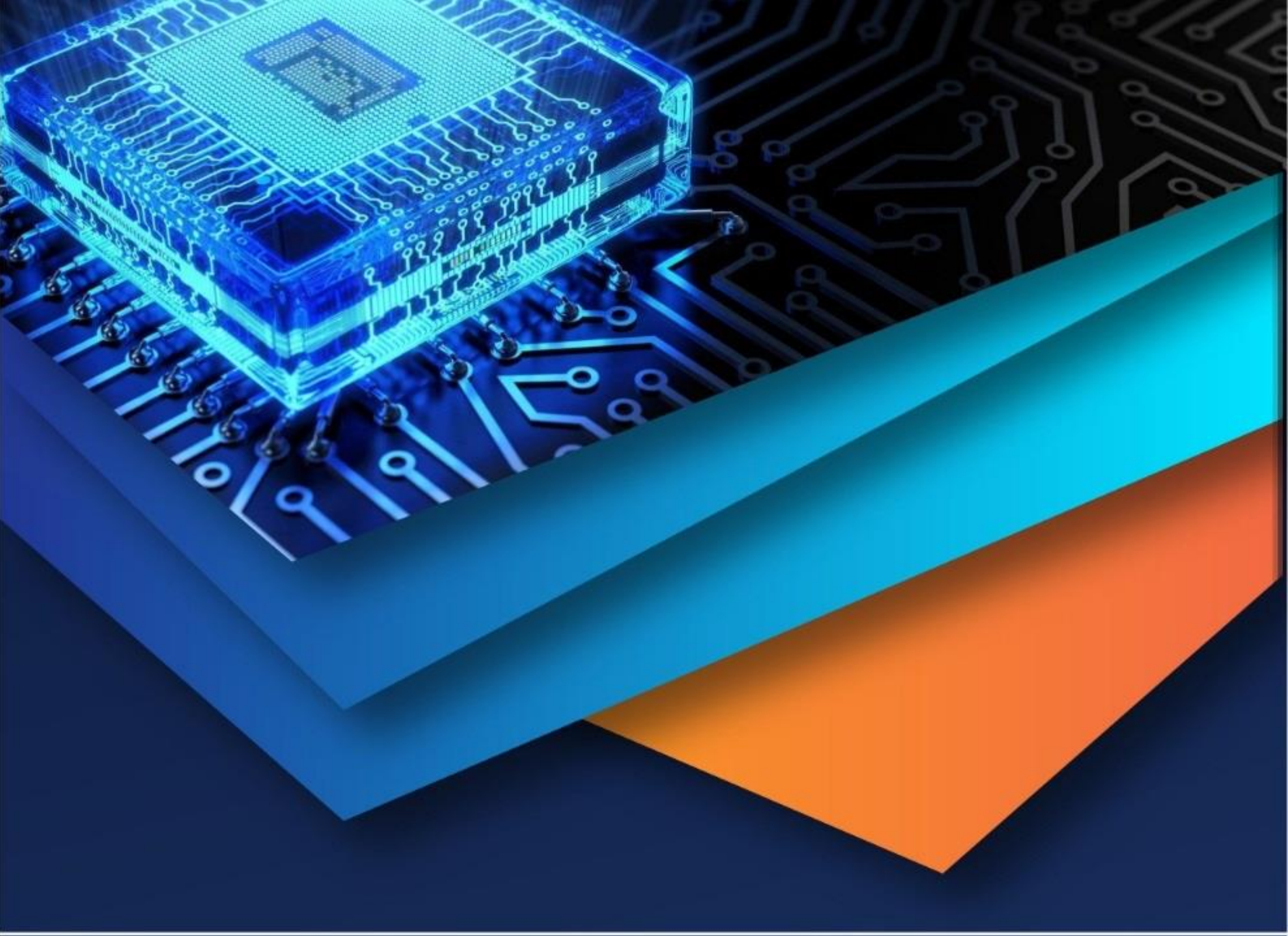
It is calculated that the eigen values of A_s *and* J_s *are* $20.1458, 6.2203, -10.3661$ *and* $0.012108, -4.195313, 4.183205$ *respectively. And hence, the matrix* A_s *is not positive definite and the eigen values of* J_s *are not less than unity in magnitude.*

IV. CONCLUISON

As seen in the above tabulated results that the Jacobi, Gauss-Seidel and SOR methods works well as long as the matrix A of (1.4) is positive definite and it is also observed from example(3.2) that all the methods discussed in this paper diverged if A of(1.4) is not positive definite.

REFERENCES

[1] Kuo.M. Solution of nonlinear equations. Computers, IEEE Transactions on,C-17(9):897-898,sep.1968.
[2] Porsching.T.A. Jacobi and gauss-seidel methods for nonlinear network problems. 6(3),1969.
[3] Richard.J.D.F.L. Burden. Numerical Analysis.8,2005.
[4] Robert.W.H. Numerical Methods. Quantum,1975.
[5] Saeed.N.A, Bhatti.A. Numerical analysis.Shahryar,2008.
[6] Varga, R.S Matrix Iterative Analysis, Prentice Hall, Englewood Cliffs NJ, 1962.
[7] Vatti.V.B.K.,Numerical analysis: Iterative MethodsI.K.International Publishing House Pvt.Ltd.(2016).
[8] Young, D.M., Linear Solution of Large Linear Systems, Academic Press, New York and London, 1971.



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