International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 9 **Issue:** VII **Month of publication:** July 2021

DOI: https://doi.org/10.22214/ijraset.2021.36926

RP-179: Formulation of Solutions of Standard Bi-Quadratic Congruence Modulo a POSITIVE INTEGER MULTIPLE to nth Power of Four

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Abstract: In this paper, the author has formulated the solutions of the standard bi-quadratic congruence of an even composite modulus modulo a positive integer multiple to nth power of four. First time a formula is established for the solutions. No literature is available for the current congruence. The author analysed the formulation of solutions in two different cases. In the first case of analysis, the congruence has the formulation which gives exactly eight incongruence solutions while in the second case of the analysis, the congruence has a different formulation of solutions and gives thirty-two incongruent solutions. A very simple and easy formulation to find all the solutions is presented here. Formulation is the merit of the paper.
Keywords: Bi-quadratic congruence, Binomial Expansion Formula, Incongruent Solutions.

I. INTRODUCTION

Congruence of the type: $x^4 \equiv b \ (mod\ m)$, m being a positive integer, is called a standard bi-quadratic congruence of prime modulus, if m is a positive odd prime integer; it is called a congruence of composite modulus, if m is a composite positive integer.

1) *Solvable Congruence:* If a congruence has a solution, then it is called solvable.
2) *Residue of an Integer:* If a positive integer b is divided by a non-zero positive integer m, then the possible remainders are called the residue of m.
3) *Bi-quadratic Residue:* A bi-quadratic residue is defined as: if r is a residue of m, then r^4 is called a bi-quadratic residue of m.

If the congruence is solvable, then $b \equiv a^4 \ (mod\ m), a\ being\ the\ residue\ of\ m$ and the congruence then can be written as: $x^4 \equiv a^4 \ (mod\ m)$. In this case, b is called bi-quadratic residue of m.

II. PROBLEM-STATEMENT

Here the problem of the study is-
"To formulate the solutions of the congruence:
$x^4 \equiv a^4 \ (mod\ b.4^n), b\ being\ a\ positive\ integer, b \not\equiv 0\ (mod\ 4)\ in\ two$
$different\ cases$".

III. LITERATURE REVIEW

Making a review through the books of Number theory [1], [2], [3] and other related literatures, the problem is found yet unformulated. Actually, a very little literature is found about the standard bi-quadratic congruence of prime and composite modulus. The author already has formulated many standard bi-quadratic congruence of composite and prime modulus [4], [5], [6], [7], [8], [9]. But the current problem was remained unnoticed. Now the author considered it for the formulation of its solutions and his efforts are presented here.

IV. ANALYSIS & RESULTS

Consider the congruence of the problem of the paper: $x^4 \equiv a^4 \ (mod\ b.4^n)$;
$b \not\equiv 0\ (mod\ 4)$.

1) **Case-I**: Let a be an odd positive integer.

For the solutions, consider $x \equiv b.4^{n-1}k \pm a \ (mod\ b.4^n)$.

Then using binomial expansion,

$$x^4 \equiv (b.4^{n-1}k \pm a)^4 \ (mod\ b.4^n)$$

$$\equiv (b.4^{n-1}k)^4 \pm 4.(b.4^{n-1}k)^3.a + \frac{4.3}{2.1}.(b.4^{n-1}k)^2.a^2 \pm \frac{4.3.2}{3.2.1}.b.4^{n-1}k.a^3 + a^4 \ (mod\ b.4^n)$$

$\equiv b^4.4^{4n-4}k^4 \pm 4.b^3.4^{3n-3}k^3.a + \frac{4.3}{2.1}.b^2 4^{2n-2}k^2a^2 \pm \frac{4.3.2}{3.2.1}.b4^{n-1}k.a^3 + a^4 \ (mod\ b.4^n)$

$\equiv b.4^n k\left\{b^3 4^{3n-4}k^3 \pm b^2 4^{2n-2}k^2.a + \frac{3}{2}.b4^{n-1}k.a^2 \pm a^3\right\} + a^4 \ (modb.4^n.$

$\equiv 0 + a^4 \ (mod\ b.4^n)$

$\equiv a^4 (mod\ b.4^n).$

So, $x \equiv b.4^{n-1}k \pm a \ (mod\ b.4^n)$ satisfies the bi-quadratic congruence and can be considered as the solutions formula for different values of k.

But for$k = 4$, the solutions formula reduces to the form:

$x \equiv b.4^{n-1}.4 \pm a \ (mod\ b.4^n).$

$\equiv b.4^n \pm a \ (mod\ b.4^n)$

$\equiv 0 \pm a \ (mod\ b.4^n).$

This is the same solutions as for $k = 0$.

Also for $k = 5 = 4 + 1$, the solutions formula reduces to the form:

$x \equiv b.4^{n-1}.(4 + 1) \pm a \ (mod\ b.4^n).$

$\equiv b.4^n + b4^{n-1} \pm a \ (mod\ b.4^n)$

$\equiv 0 + b.4^{n-1} \pm a \ (mod\ b.4^n).$

$\equiv b.4^{n-1} \pm a \ (mod\ b.4^n).$

This is the same solutions as for $k = 1$.

Therefore, $x \equiv b.4^{n-1}k \pm a \ (mod\ b.4^n)$ gives all the solutions of the said congruence when a is an odd positive integer with k=0, 1, 2, 3.

This gives exactly eight incongruent solutions of the congruence.

2) *Case-II:* Let a be an even positive integer.

For the solutions, consider $x \equiv b.4^{n-2}k \pm a \ (mod\ b.4^n)$.

Then using binomial expansion,

$x^4 \equiv (b.4^{n-2}k \pm a)^4 \ (mod\ b.4^n)$

$\equiv (b.4^{n-2}k)^4 \pm 4.(b.4^{n-2}k)^3.a + \frac{4.3}{2.1}.(b.4^{n-2}k)^2.a^2 \pm \frac{4.3.2}{3.2.1}.b.4^{n-2}k.a^3 + a^4 \ (mod\ b.4^n)$

$\equiv b^4.4^{4n-8}k^4 \pm 4.b^3.4^{3n-6}k^3.a + \frac{4.3}{2.1}.b^2 4^{2n-4}k^2a^2 \pm \frac{4.3.2}{3.2.1}.b4^{n-2}k.a^3 + a^4 \ (mod\ b.4^n)$

$\equiv b.4^{n-1}k\left\{b^3 4^{3n-4}k^3 \pm b^2 4^{2n-2}k^2.a + \frac{3}{2}.b4^{n-1}k.a^2 \pm a^3\right\} + a^4 \ (mod\ b.4^n).$

$\equiv b.4^{n-1}k\{4t\} + a^4 \ (mod\ b.4^n), if\ b\ is\ even\ positive\ integer.$

$\equiv 0 + a^4 \ (mod\ b.4^n)$

$\equiv a^4 (mod\ b.4^n).$

So, $x \equiv b.4^{n-2}k \pm a \ (mod\ b.4^n)$ satisfies the bi-quadratic congruence and can be considered as the solutions formula for different values of k.

But for$k = 16 = 4^2$, the solutions formula reduces to the form:

$x \equiv b.4^{n-2}.4^2 \pm a \ (mod\ b.4^n).$

$\equiv b.4^n \pm a \ (mod\ b.4^n)$

$\equiv 0 \pm a \ (mod\ b.4^n).$

This is the same solutions as for $k = 0$.

Also for $k = 17 = 4^2 + 1$, the solutions formula reduces to the form:

$x \equiv b.4^{n-2}.(4^2 + 1) \pm a \ (mod\ b.4^n).$

$\equiv b.4^n + b4^{n-2} \pm a \ (mod\ b.4^n)$

$\equiv 0 + b.4^{n-2} \pm a \ (mod\ b.4^n).$

$\equiv b.4^{n-2} \pm a \ (mod\ b.4^n).$

This is the same solutions as for $k = 1$.

Therefore, $x \equiv b.4^{n-2}k \pm a \ (mod\ b.4^n)$ gives all the solutions of the said congruence when a is an odd positive integer with k=0, 1, 2, 3, ……….., 15.

This gives exactly thirty-two incongruent solutions of the congruence.

V. ILLUSTRATIONS

1) *Example-1:* Consider the congruence $x^4 \equiv 81 \ (mod\ 512)$.

It can be written as $x^4 \equiv \ 3^4 \ (mod\ 2.4^4)$.

It is of the type $x^4 \equiv \ a^4 \ (mod\ b.4^n), with\ b = 2, n = 4, a = 3, a\ positive\ integer$.

It has exactly 8 incongruent solutions given by

$$x \equiv b.4^{n-1}k \pm a \ (mod\ b.4^n)\ with\ k = 0,1,2,3.$$

$$\equiv 2.4^3k \pm 3 \ (mod\ 2.4^4)$$

$$\equiv 128k \pm 3 \ (mod\ 512)$$

$$\equiv 0 \pm 3; 128 \pm 3; 256 \pm 3; 384 \pm 3 \ (mod\ 512)$$

$$\equiv 3, 509; 125, 131; 253, 259; 381, 387 \ (mod\ 512).$$

2) *Example-2:* Consider the congruence $x^4 \equiv 1296 \ (mod\ 1536)$.
It can be written as $x^4 \equiv \ 6^4 \ (mod\ 6.4^4)$.

It is of the type $x^4 \equiv \ a^4 \ (mod\ b.4^n), with\ b = 6, n = 4, a = 6, a\ positive\ integer$.

It has exactly 32 incongruent solutions given by

$$x \equiv b.4^{n-2}k \pm a \ (mod\ b.4^n)\ with\ k = 0,1,2,3, \ldots \ldots . . 4^2 - 1.$$

$$\equiv 6.4^2k \pm 6 \ (mod\ 6.4^4)\ with\ k = 0, 1, 2, 3, \ldots \ldots \ldots . . , 15.$$

$$\equiv 96k \pm 6 \ (mod\ 1536)$$

$$\equiv 0 \pm 6; 96 \pm 6; 192 \pm 6; 288 \pm 6; 384 \pm 6; 480 \pm 6; 576 \pm 6; 672 \pm 6;$$

$$768 \pm 6; 864 \pm 6; 960 \pm 6; 1056 \pm 6; 1152 \pm 6; 1248 \pm 6; 1344 \pm 6;$$

$$1440 \pm \ 6 \ (mod\ 1536)$$

$$\equiv 6, 1530; 90, 102; 186, 198; 282, 294; 378, 390; \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$1338, 1350; 1434, 1446 \ (mod\ 1536).$$

3) *Example-3:* Consider the congruence $x^4 \equiv 625 \ (mod\ 1792)$.
It can be written as $x^4 \equiv \ 5^4 \ (mod\ 7.4^4)$.
It is of the type $x^4 \equiv \ p^4 \ (mod\ p.4^n)\ with\ a = p = 5, n = 4$.
It has exactly $2.4 = 8$ incongruent solutions given by

$$x \equiv p.4^{n-1}k \pm p \ (mod\ p.4^n)\ with\ k = 0,1,2,3.$$

$$\equiv 5.4^3k \pm 5 \ (mod\ 5.4^4)$$

$$\equiv 5.64k \pm 5 \ (mod\ 1280)$$

$$\equiv 320k \pm 5 \ (mod\ 1280)$$

$$\equiv 0 \pm 5; 320 \pm 5; 640 \pm 5; 960 \pm 5 \ (mod\ 1280)$$

$$\equiv 5,1275; 315, 325; 635, 645; 955, 965 \ (mod\ 1280).$$

4) *Example-4:* Consider the congruence $x^4 \equiv 609\ (mod\ 1792)$.
It can be written as $x^4 \equiv 609 + 1792 = 2401 = 7^4\ (mod\ 7.4^4)$.

It is of the type $x^4 \equiv p^4\ (mod\ p.4^n)\ with\ a = p = 7, n = 4$.

It has exactly $2.4 = 8$ incongruent solutions given by

$$x \equiv p.4^{n-1}k \pm p\ (mod\ p.4^n)\ with\ k = 0,1,2,3.$$

$$\equiv 7.4^3k \pm 7\ (mod\ 7.4^4)$$

$$\equiv 7.64k \pm 7\ (mod\ 1792)$$

$$\equiv 448k \pm 7\ (mod\ 1792)$$

$$\equiv 0 \pm 7; 448 \pm 7; 896 \pm 7; 1344 \pm 7\ (mod\ 1792)$$

$$\equiv 7,1785; 441,455; 889,903; 1337,1351\ (mod\ 1792).$$

VI. CONCLUSION

Therefore it is concluded that the standard bi-quadratic congruence: $x^4 \equiv a^4\ (mod\ b.4^n)$, b a positive integer, has exactly eight solutions: $x \equiv b.4^{n-1}k \pm a\ (mod\ b.4^n)\ with$
$k = 0,1,2,3$, if a is an odd positive integer but the congruence has exactly thirty-two incongruent solutions: $x \equiv b.4^{n-2}k \pm a\ (mod\ b.4^n)\ with\ k = 0,1,2,3,\ldots\ldots..(32-1)$,
$if\ a$ is an even positive integer.

VII. MERIT OF THE PAPER

Due to the discovery of the formulation, it became possible to find the solutions easily. The solutions can also be obtained orally. This is the merit of the paper.

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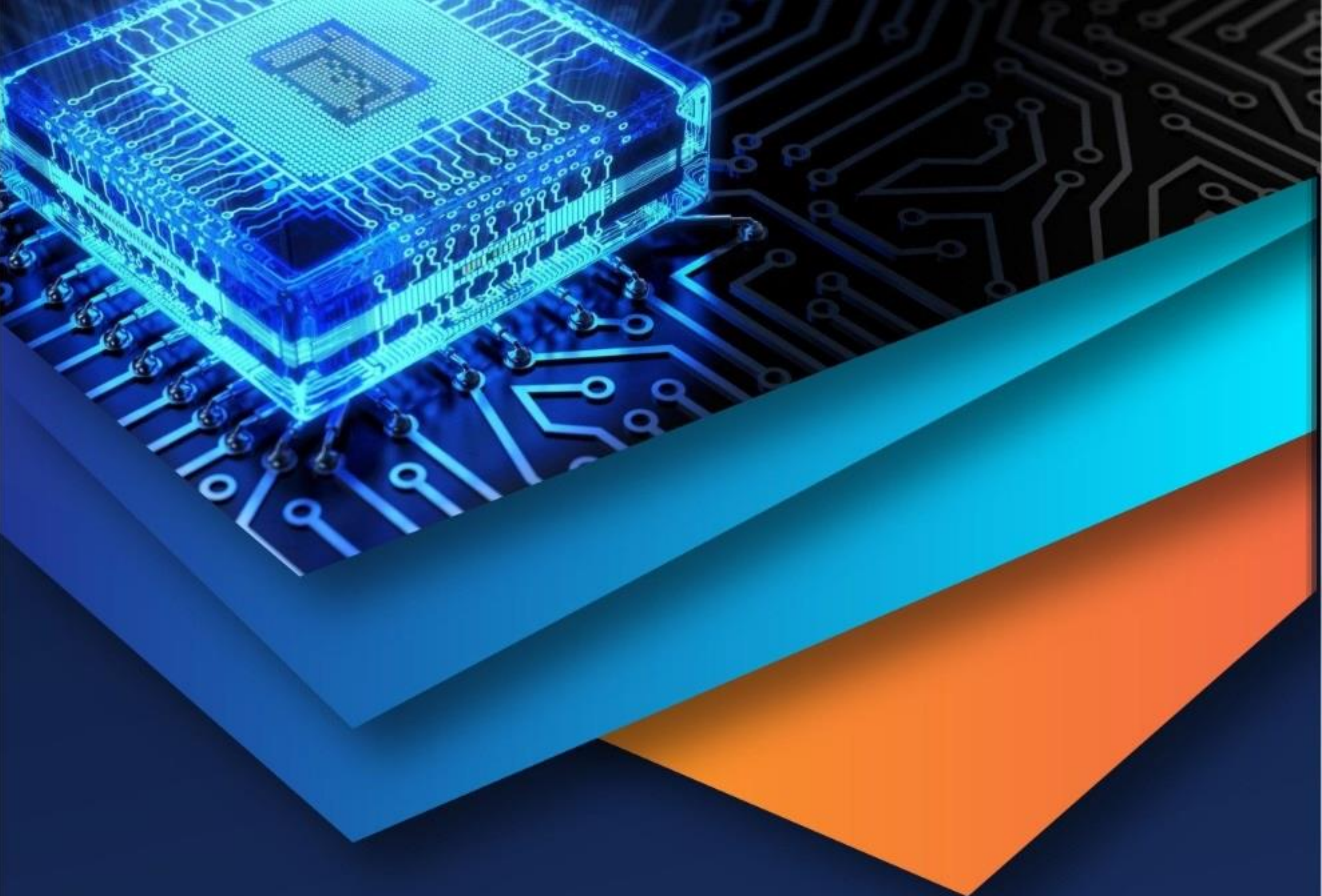

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