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Integral Representation of Polynomial $X_n(x; a, b)$

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Abstract: In the present paper we have obtained fine difference formula, contour integral representation, real integral representation, infinite single integral representation, finite single integral representation, finite double integral representation, finite double integral representation of polynomial $X_n(x; a, b)$.
Keywords: Finite difference, single integral representation, contour integral representation, simple generating relation, double integral representation

I. INTRODUCTION

Bajpai, S.D.[1,1993] defined the classical polynomial

$$X_n(x; a, \propto) = {}_2F_1\left[-n, a; -\frac{x}{\propto}\right] \tag{1}$$

II. FINITE DIFFERENCE FORMULA

From (1) $\quad X_n(x; a, \propto) = {}_2F_1\left[-n, a; -\frac{x}{\propto}\right]$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(\frac{x}{\propto}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (-n)_k (a)_k \left(\frac{x}{\propto}\right)^k}{k!}$$

But $\quad (-n)_k = \frac{(-1)^k n!}{(n-k)!} \quad (0 \le k \le n)$

$= 0 \qquad k > n$

$$X_n(x; a, \propto) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(a)_k \left(-\frac{x}{\propto}\right)^k}{k!} \tag{2}$$

$$X_n(x; a, \propto) = \sum_{k=0}^{n} \frac{(-1)^n (-1)^{n-k} n!}{(n-k)!} \frac{(a)_k \left(-\frac{x}{\propto}\right)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^n (-1)^{n-k} n!}{(n-k)!} \frac{\Gamma(a+k) \left(-\frac{x}{\propto}\right)^k}{\Gamma a \; k!}$$

$$= \frac{(-1)^n}{\Gamma a} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \Gamma(a+k) \left(-\frac{x}{\propto}\right)^k$$

Replacing a by a+λ we get

$$X_n(x; a+\lambda, \propto) = \frac{(-1)^n}{\Gamma a} \left(\frac{x}{\propto}\right)^{-\lambda} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \Gamma(a+k) \left(-\frac{x}{\propto}\right)^{k+\lambda}$$

$$= \frac{(-1)^n}{\Gamma(a+\lambda)}\left(\frac{x}{\propto}\right)^{-\lambda}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\ \Gamma(a+\lambda+k)\ \left(-\frac{x}{\propto}\right)^{k+\lambda}$$

$$= \frac{(-1)^n}{\Gamma(a+\lambda)}\left(\frac{x}{\propto}\right)^{-\lambda}\ \Delta_\lambda^n\left[\Gamma(a+\lambda)\ \left(-\frac{x}{\propto}\right)^{k+\lambda}\right]$$

Hence we have proved that

$$X_n(x; a+\lambda, \propto) = \frac{(-1)^n}{\Gamma(a+\lambda)}\left(\frac{x}{\propto}\right)^{-\lambda}\ \Delta_\lambda^n\left[\Gamma(a+\lambda)\ \left(-\frac{x}{\propto}\right)^{k+\lambda}\right] \quad (3)$$

III. SIMPLE GENERATING RELATION

A. By equation (2)

$$\sum_{n=0}^{\infty}\frac{X_n(x;a,\propto)t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k\, n!}{(n-k)!}\frac{(a)_k\left(-\frac{x}{\propto}\right)^k}{k!}\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}\frac{X_n(x;a,\propto)t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n)!}\frac{(a)_k\left(-\frac{x}{\propto}\right)^k}{k!}\frac{t^{n+k}}{}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{}\frac{(a)_k\left(-\frac{xt}{\propto}\right)^k}{k!}\frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{}\frac{(a)_k\left(-\frac{xt}{\propto}\right)^k}{k!}\sum_{k=0}^{\infty}\frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{}\frac{(a)_k\left(-\frac{xt}{\propto}\right)^k}{k!}e^t$$

$$= e^t\sum_{k=0}^{\infty}\frac{(a)_k\left(\frac{xt}{\propto}\right)^k}{k!}$$

$$= e^t\,{}_1F_0\left[a; -; \frac{xt}{\propto}\right]$$

$$f(t) = e^t\,{}_1F_0\left[a; -; \frac{xt}{\propto}\right]$$

B. By using Maclaurin's theorem

$$f(t) = \sum_{k=0}^{\infty}\frac{f^{(n)}(0)\ t^n}{n!}$$

$$f^n(0) = \frac{n!}{2\pi i}\int^{(+0)}\frac{f(t)}{t^{n+1}}\,dt \quad \forall\ n = 0,1,2,\ldots$$

If $e^t\,{}_1F_0\left[a; -; \frac{xt}{\propto}\right] = \sum_{n=0}^{\infty}\frac{X_n(x;a,b)t^n}{n!}$ *,then*

$$X_n(x; a, \propto) = \frac{n!}{2\pi i}\int^{(+0)} t^{-n-1}\,e^t\ {}_1F_0\left[a; -; \frac{xt}{\propto}\right]\ dt \quad (4)$$

IV. REAL INTEGRAL REPRESENTATION

Put $t = e^{i\theta} \quad (0 \le \theta \le 2\pi) \quad in\ (4)$

$$dt = ie^{i\theta}\, d\theta$$

$$X_n(x;a,\propto) = \frac{n!}{2\pi i} \int_0^{2\pi} \left(e^{i\theta}\right)^{-n-1} \exp(e^{i\theta}) \quad {}_1F_0\left[a;-;\frac{xe^{i\theta}}{\propto}\right] \quad ie^{i\theta} d\theta$$

$$X_n(x;a,\propto) = \frac{n!}{2\pi} \int_0^{2\pi} \left(e^{-in\theta}\right) \exp(e^{i\theta}) \quad \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \left(\frac{xe^{i\theta}}{\propto}\right)^k d\theta$$

$$= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \int_0^{2\pi} \left(e^{i(k-n)\theta}\right) \exp(e^{i\theta}) \quad \left(\frac{x}{\propto}\right)^k d\theta$$

$$= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \int_0^{2\pi} \left(e^{i(k-n)\theta}\right) \sum_{s=0}^{\infty} \frac{\left(e^{i\theta}\right)^s}{s!} \quad \left(\frac{x}{\propto}\right)^k d\theta$$

$$= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{s!} \frac{(a)_k}{k!} \left(\frac{x}{\propto}\right)^k \int_0^{2\pi} \left(e^{i(k+s-n)\theta}\right) \quad d\theta$$

$$= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{s!} \frac{(a)_k}{k!} \left(\frac{x}{\propto}\right)^k \int_0^{2\pi} Cis(k+s-n)\theta \quad d\theta$$

where $Cis\varphi = \cos\varphi + i\sin\varphi$

$$= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_k}{k!\ s!} \left(\frac{x}{\propto}\right)^k \int_0^{2\pi} [\cos\varphi + i\sin\varphi] \quad d\theta$$

where $\varphi = (k+s-n)\theta$

$$X_n(x;a,\propto) = \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_k}{k!\ s!} \left(\frac{x}{\propto}\right)^k \int_0^{2\pi} [\cos\varphi + i\sin\varphi] \quad d\theta \tag{5}$$

where $\varphi = (k+s-n)\theta.$

V. SINGLE INFINTE INTEGRAL REPRESENTATION

From (1) $X_n(x;a,\alpha) = {}_2F_1\left[-n,b;-\frac{x}{\propto}\right]$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(-\frac{x}{\propto}\right)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{\Gamma(a+k)\ \left(-\frac{x}{\propto}\right)^k}{\Gamma a}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k}{k!\ \Gamma a} \frac{\Gamma\left(a+k+\frac{1}{2}\right)-\frac{1}{2}\ \left(-\frac{x}{\propto}\right)^k}{}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k}{k!\ \Gamma a} \left(-\frac{x}{\propto}\right)^k \int_{-\infty}^{\infty} exp(-t^2)\, t^{2\left(a+k-\frac{1}{2}\right)} dt$$

$$= \frac{1}{\Gamma a} \int_{-\infty}^{\infty} exp(-t^2)\, t^{2a-1} \sum_{k=0}^{n} \frac{(-n)_k}{k!} \left(-\frac{xt^2}{\propto}\right)^k dt$$

$$= \frac{1}{\Gamma a} \int_{-\infty}^{\infty} exp(-t^2)\, t^{2a-1}\ {}_1F_0\left[-n;-;-\frac{xt^2}{\propto}\right] dt \tag{6}$$

VI. FINITE DOUBLE INTEGRAL REPRESENTATION

Srivastava, H.M. and Karlsson, P.W.[5, P.275]

$$\iint_D u^{a-1} v^{b-1} (1-u-v)^{c-1} du\, dv = \frac{\Gamma a\ \Gamma b\ \Gamma c}{\Gamma(a+b+c)}$$

Where D is bounded by the lines $u \geq 0, v \geq 0$ *and* $u + v \leq 1.$

$$X_n(x; a, c) = {}_2F_1\left[-n, b;\ -\frac{x}{c}\right]$$

$$X_n(x; a, c) = \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(\frac{x}{c}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k\, \Gamma(a+k)\ \left(-\frac{x}{c}\right)^k}{\Gamma a\ \ k!}$$

$$= \frac{\Gamma\alpha}{\Gamma a \Gamma b\ \Gamma(\alpha-a-b)} \sum_{k=0}^{\infty} \frac{(-n)_k\, \Gamma b\ \Gamma(\alpha-a-b)\ \Gamma(a+k)\ (\alpha)_k\ \left(-\frac{x}{c}\right)^k}{\Gamma(\alpha+k)\ k!}$$

$$= \frac{\Gamma\alpha}{\Gamma a \Gamma b\ \Gamma(\alpha-a-b)} \sum_{k=0}^{\infty} \frac{(-n)_k (\alpha)_k}{k!} \iint_D u^{a+k-1} v^{b-1} (1-u-v)^{\alpha-a-b-1} \left(-\frac{x}{c}\right)^k du\, dv$$

$$= \frac{\Gamma\alpha}{\Gamma a \Gamma b\ \Gamma(\alpha-a-b)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\alpha-a-b-1} \sum_{k=0}^{\infty} \frac{(-n)_k (\alpha)_k}{k!} \left(-\frac{xu}{c}\right)^k du\, dv$$

$$= \frac{\Gamma\alpha}{\Gamma a \Gamma b\ \Gamma(\alpha-a-b)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\alpha-a-b-1}\ {}_2F_1\left[-n, b; -\frac{xu}{c}\right] du\, dv \qquad (7)$$

VII. INFINITE SINGLE INTEGRAL REPRESENTATION

From equation (1), $X_n(x; a, \alpha) = {}_2F_1\left[-n, b;\ -\frac{x}{\propto}\right]$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(-\frac{x}{\propto}\right)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(a)_k \left(-\frac{x}{\propto}\right)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k}{(k)!} \frac{\Gamma(a+k)\ \left(-\frac{x}{\propto}\right)^k}{\Gamma a}$$

$$= \frac{1}{\Gamma a} \int_0^{\infty} \sum_{k=0}^{n} \frac{(-n)_k}{(k)!} \frac{\left(-\frac{x}{\propto}\right)^k}{} e^{-t}\, t^{a+k-1}\, dt$$

$$= \frac{1}{\Gamma a} \int_0^{\infty} \sum_{k=0}^{n} \frac{(-n)_k}{(k)!} \frac{\left(-\frac{xt}{\propto}\right)^k}{} e^{-t}\, t^{a-1}\, dt$$

$$= \frac{1}{\Gamma a} \int_0^{\infty} \left(1 + \frac{xt}{\propto}\right)^n e^{-t}\, t^{a-1}\, dt$$

$$= \frac{1}{\alpha^n \Gamma a} \int_0^{\infty} e^{-t}\, t^{a-1}\ (\alpha + xt)^n\ dt \qquad (8)$$

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VIII. INFINITE DOUBLE INTEGRAL REPRESENTATION

From [2 ,P.177(16)],

$$\int_0^\infty \int_0^\infty \varphi(x+y)x^\alpha y^\beta \, dx\, dy = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \int_0^\infty \varphi(z)\; z^{\alpha+\beta+1} \; dz$$

$$\int_0^\infty \int_0^\infty u^{\alpha+1/2}\; v^\alpha \,(1-u-v)^{-3/2} X_n(x;a,4uvc)\, du\, dv$$

$$= \int_0^\infty \int_0^\infty u^{\alpha+1/2}\; v^\alpha \,(1-u-v)^{-3/2} \sum_{k=0}^\infty \frac{(-n)_k (a)_k \left(-\frac{x}{4uvc}\right)^k}{k!} du\, dv$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k}{k!} \int_0^\infty \int_0^\infty u^{\alpha+1/2}\; v^\alpha\, 2^{-2k}\, u^{-k} v^{-k} (1-u-v)^{-3/2} \left(-\frac{x}{c}\right)^k du\, dv$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{\Gamma(2\alpha-2k+5/2)} \int_0^\infty z^{2\alpha-2k+3/2} \; (1-z)^{-3/2}\, dz$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{\Gamma(2\alpha-2k+5/2)} \beta\left(2\alpha+2k+\frac{5}{2}, 2k-2\alpha-1\right)$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{\Gamma(2\alpha-2k+5/2)} \frac{\Gamma\left(2\alpha+2k+\frac{5}{2}\right)\Gamma(2k-2\alpha-1)}{\Gamma(3/2)}$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{} \frac{\Gamma(2k-2\alpha-1)}{\Gamma(3/2)}$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+1)\Gamma\left(\alpha-k+1+\frac{1}{2}\right)\Gamma(2k-2\alpha-1)}{\frac{1}{2}\sqrt{\pi}}$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k \; 2^{-2k}}{k!} \frac{\left(-\frac{x}{c}\right)^k \sqrt{\pi}\,\Gamma(2\alpha-2k+2)\Gamma(2k-2\alpha-1)}{\frac{1}{2}\sqrt{\pi} \;\; 2^{2\alpha-2k+2-1}}$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k}{k!} \frac{\left(-\frac{x}{c}\right)^k \Gamma(2\alpha-2k+2)\Gamma(2k-2\alpha-1)}{2^{2\alpha}}$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k}{k!} \frac{\left(-\frac{x}{c}\right)^k \Gamma(2\alpha-2k+2)\Gamma(1-(2\alpha-2k+2))}{2^{2\alpha}}$$

$$= \sum_{k=0}^\infty \frac{(-n)_k (a)_k \left(-\frac{x}{c}\right)^k}{k! \;\; 2^{2\alpha}} \; \frac{\pi}{\sin[\pi(2\alpha-2k+2)]}$$

$$= \frac{\pi}{2^{2\alpha}\; \sin(2\pi\alpha)} \sum_{k=0}^\infty \frac{(-n)_k (a)_k \left(-\frac{x}{c}\right)^k}{k!}$$

$$= \frac{\pi}{2^{2\alpha}\; \sin(2\pi\alpha)} \qquad X_n(x;a,c)$$

Thus we arrive at

$$X_n(x;a,c) = \frac{2^{2\alpha}\; \sin(2\pi\alpha)}{\pi} \int_0^\infty \int_0^\infty u^{\alpha+1/2}\; v^\alpha\,(1-u-v)^{-3/2} X_n(x;a,4uvc)\, du\, dv \qquad (9)$$

The equations (4), (5), (6), (7), (8), (9) *are not in literature.*

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