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Analytical Solution for Boltzmann Collision Operator for the1-D Diffusion equation

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Abstract: In this paper, we have presented the analytical solution of the collision operator for the Boltzmann equation of one-dimensional diffusion equation using the analytical solution of the one-dimensional Navier Stoke diffusion equation.
Keywords: Boltzmann equation; analytical collision operator; one-dimensional diffusion equation.

I. INTRODUCTION

The relation between the macroscopic states of a system and microscopic states at equilibrium has been established by Boltzmann distribution function:

$$f^i = \frac{1}{z} e^{\frac{-\epsilon_i}{KT}} \qquad (1)$$

Where f^i the probability of an atom to be in i space phase cell, z is the partition function, ϵ_i the energy of the i space phase cell, K is the Boltzmann constant, T is the temperature.

Boltzmann was first to realize the connection between the entropy and the possible microscopic states of the system, a fundamental idea leads to arise a new field called statistical mechanic, the Boltzmann equation arose from a very fundamental question how this equilibrium will be reached, in 1872 the Boltzmann equation invented by Boltzmann to describe the evolving of generalized Boltzmann- Maxwell distribution with the time:

$$\frac{\partial f}{\partial t} + c \cdot \nabla f + \frac{F}{m} \cdot \frac{\partial f}{\partial c} = \Omega \qquad (2)$$

Where the first term on the left-hand side describes the changing of the distribution function f with the time t, the second term represents the changing of the distribution function with spatial space, it is an advection term with the effect of the velocity c, the third term represents the changing of the distribution function with the velocity as well as the force F acting on it divided by mass m. On the right-hand side, the collision operator Ω.

The collision term of the Boltzmann equation is crucial to solve Boltzmann equation, Boltzmann introduced and determined the collision term according to his assumption known as a molecular chaos assumption, the form of this collision operator takes double integral over spatial and velocity space, it is complicated and time consuming, this collision term is only valid for gas simulation because it has been derived according to the binary collision assumption.

Due to the abovementioned challenges many attempts to simplify and generalize the collision operator, one of the most widely used to solve the collision operator of the Boltzmann equation is BGK approximation invented by Bhatnagar, Gross and Krook in 1954, the popularity of the BGK came from the simplicity of the mathematical expression of the collision operator:

$$\Omega = -\frac{f - f^{eq}}{\tau} \qquad (3)$$

Where f^{eq} represents the distribution function in equilibrium state, τ represents the time needed for the distribution function in non-equilibrium state to achieve the equilibrium state called relaxation time.

But what is the real accurate expression of the collision operator, in another word what is the analytical expression of the collision operator, it is well known the generic mathematical expression is difficult to be found due to the varieties and complexities of the different phenomena, since the analytical solution of the diffusion equation exist, we can derive the analytical solution of the collision operator by using it by finding the expression of the collision operator we can use it to compare it between other collision expressions for the same type of the phenomena.

II. 1D -HEAT DIFFUSION AND BOLTZMANN EQUATIONS

The mathematical equation to describe the heat distribution along one dimension is:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{4}$$

Where T is the temperature, k is the thermal diffusivity, t is the time, x is a spatial dimension, we can modify the shape of the equation (4) by the following assumption:

$$T = u(t) \times e^{\int R(x)dx} \tag{5}$$

Where u(t) is unknown function of one independent variable t, R(x) is unknown function of one independent variable x, substitute equation (5) in (4) leads to:

$$\frac{u''(t)}{u(t)} - kR'(x) = k[R(x)]^2 \tag{6}$$

The 1D-Boltzmann equation with absence of the force can be written as:

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = \Omega \tag{7}$$

By comparing equation (6) and (7) we can easily equate the first term of the equation (7) with the first term of the equation (6) and so on, this will lead to:

$$f = lnu(t) + A(x) \tag{8}$$

$$f = -\frac{k}{c}R(x) + B(t) \tag{9}$$

$$\Omega = k[R(x)]^2 \tag{10}$$

Where A(x) and B(t) are the unknown functions of x and t respectively, resulted from the integration of the partial derivative with respect to t and x respectively, by equating the equation (8) and (9) we can find the unknown functions as follows:

$$A(x) = -\frac{k}{c}R(x) \tag{11}$$

$$B(t) = lnu(t) \tag{12}$$

So f can be written as:

$$f = lnu(t) - \frac{k}{c}R(x) \tag{13}$$

Equation (13) satisfy the following condition:

$$\frac{df}{dt} = \Omega \tag{14}$$

By substituting equation (13) in (14) leads to:

$$\frac{u''(t)}{u(t)} - \frac{k}{c}R'(x)\frac{dx}{dt} = \Omega \tag{15}$$

But dx/dt is c, equation (15) becomes:

$$\frac{u''(t)}{u(t)} - kR'(x) = \Omega \tag{15}$$

Equation (16) satisfies equation (6).
The macroscopic parameter of Boltzmann equation can be covered according to this equation:

$$T = \int f dc \tag{17}$$

If we sub equation (13) in (5) and (10) we get:

$$\Omega = \frac{c^2}{k}[lnu(t) - f]^2 \tag{18}$$

$$T = e^{f + \frac{k}{c}R(x) + \int R(x) dx} \tag{19}$$

III. Analytical Solution Of The Collision Operator

We can derive the solution of the collision operator form the solution of the heat equation, we have set the solution of heat diffusion by assuming the temperature as a function of the time u(t) multiplied by the function $e^{\int R(x)dx}$ of one spatial dimension x, the solution of heat diffusion depends on the boundary conditions, if we assume we have one dimension rod subjected to the following boundary condition:

$$T(x,0) = z(x) \quad \forall x \in [0, L] \tag{20}$$

$$T(0,t) = T(L,t) \quad \forall t > 0 \tag{21}$$

Where the z(x) represents the distribution of the temperature along the rod and L is the length of the rod.
The solution of the heat equation with this boundary conditions is called separation of variable and we can use it for finding the unknown functions u(t) and R(x) as follows:

$$u_n(t) = e^{\frac{n^2\pi^2 k}{L}t} \tag{22}$$

$$e^{\int R(x)_n dx} = D_n \sin\left(\frac{n\pi x}{L}\right) \tag{23}$$

$$D_n = \frac{2}{L}\int_0^L z(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{24}$$

Where n is the positive integer number, D_n is the coefficient needs to be calculated according to equation (24), form equation (22) and (23) we can find the following:

$$\ln u_n(t) = \frac{n^2\pi^2 k}{L}t \tag{25}$$

$$R(x)_n = \frac{n\pi}{L}\tan\left(\frac{n\pi}{L}x\right) \tag{26}$$

Sub equation (25) and (26) in (13) we get:

$$f_n = \frac{n^2\pi^2 k}{L}t - \frac{k}{c}\frac{n\pi}{L}\tan\left(\frac{n\pi}{L}x\right) \tag{27}$$

Sub equation (23) and (26) in (17) leads to:

$$\int f_n\, dc = e^{f_n + \frac{kn\pi}{c\,L}\tan\left(\frac{n\pi}{L}x\right)} D_n \sin\left(\frac{n\pi x}{L}\right) \tag{28}$$

Sub equation (25) in (18) leads to:

$$\Omega_n = \frac{c^2}{k}\left[\frac{n^2\pi^2 k}{L}t - f_n\right]^2 \tag{29}$$

The solution of the Boltzmann equation for this problem will be the exact solution of the heat diffusion equation by separation of variable:

$$\sum_{n=0}^{\infty}\left[\frac{\partial f_n}{\partial t} + c\frac{\partial f_n}{\partial x}\right] = \sum_{n=0}^{\infty}\Omega_n \tag{30}$$

$$T = \sum_{n=0}^{\infty} e^{f_n + \frac{kn\pi}{c\,L}\tan\left(\frac{n\pi}{L}x\right)} D_n \sin\left(\frac{n\pi x}{L}\right) \tag{31}$$

IV. CONCLUSIONS

Equations (18) and (29) show that the collision operator is a quadratic function with two variables t and f in which the highest degree term is of the second order, the exact solution of the collision operator gives an ideation about the collision operator how it should be approximated and open a new imagination how the collision operator should be dealt with it, if we expand the equation (18) we find out that the variable time is implicitly impeded with variable f, we may assume the collision operator has a generic form of $\Omega = Af^2 + Bf + c$, finding the coefficients A, B and C is a challenging problem since its highly depended on the type of the phenomena and the boundary conditions for each phenomenon has.

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