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An analysis on Quantum mechanical Stability of Regular Polygons on a Point Base Using Heisenberg Uncertainty Principle

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***Abstract:** It is a well-known fact in physics that classical mechanics describes the macro-world, and quantum mechanics describes the atomic and sub-atomic world. However, principles of quantum mechanics, such as Heisenberg's Uncertainty Principle, can create visible real-life effects. One of the most commonly known of those effects is the stability problem, whereby a one-dimensional point base object in a gravity environment cannot remain stable beyond a time frame. This paper expands the stability question from 1- dimensional rod to 2-dimensional highly symmetrical structures, such as an even-sided polygon. Using principles of classical mechanics, and Heisenberg's uncertainty principle, a stability equation is derived. The stability problem is discussed both quantitatively as well as qualitatively. Using the graphical analysis of the result, the relation between stability time and the number of sides of polygon is determined. In an environment with gravity forces only existing, it is determined that stability increases with the number of sides of a polygon. Using the equation to find results for circles, it was found that a circle has the highest degree of stability. These results and the numerical calculation can be utilized for architectural purposes and high-precision experiments. The result is also helpful for minimizing the perception that quantum mechanical effects have no visible effects other than in the atomic, and subatomic world.*

***Keywords:** Quantum mechanics, Heisenberg Uncertainty principle, degree of stability, polygon, the highest degree of stability*

I. INTRODUCTION

The uncertainty principle is one of the cornerstones of quantum mechanics governing various phenomena on the atomic and subatomic scale. The uncertainty principle states that the more precisely the position of some particle is determined, the less precisely the momentum of the particle can be predicted from initial conditions, and vice versa. [1] The formal inequality relating the standard deviation of position σ_x and the standard deviation of momentum σ_p was derived by Earle Hesse Kennard and by Hermann Weyl in 1928. [2] [3]

$$\sigma_x.\sigma_p \geq \frac{\hbar}{2}$$

where $\hbar$ is the reduced Planck constant, $h/(2\pi)$.

This principle is very useful in nano-scales and is not readily observed in the macroscopic scales of everyday experience. [4] However, it also plays a visible effect in some classical macro world phenomenon. One particular example is a 1-d rod balancing problem in a gravity environment, also known as the pencil balancing problem. This principle can be further applied to examine 2-dimensional even-sided regular polygons for determining their stability for point base situations. For this, the classical expression on angular acceleration based on the center of mass is applied alongside the Heisenberg uncertainty principle to determine the time for stability of such systems. The general equation thus derived is graphed to generalize the result, and find the most stable structure and the extent to which a classical system can be stable quantum mechanically.

II. 1-DIMENSIONAL STABILITY PROBLEM

This problem is widely discussed and solved in undergraduate physics courses as a balancing a pencil problem. Here, the core concept is combining classical mechanics with quantum mechanics. Let us imagine a rod of length l and mass m standing on its tip on support in a gravity environment with acceleration due to gravity g. We ignore the fine structural details of the rod tip, treating it as a point. In an environment where gravity is present, its stability depends on its initial angle from the vertical as well as its momentum, or velocity of the center of mass. If both are zero, we can argue that the rod is perfectly stable. This condition is perfectly possible in classical mechanics. However, according to the Heisenberg uncertainty principle, we cannot determine both the values accurately and the values follow the inequality equation given by Earle Hesse Kennard and by Hermann Weyl.

Let us assume that the initial angular displacement from the vertical and angular velocity of the rod about the vertical are θ_0 and ω_0 respectively. We assume the mass to be concentrated on its center of mass. Let the center of mass be d distance from the tip. Let $\theta(t)$ be the displacement of the rod at any time t. Using the torque- angular acceleration relation is given below:

$$\Sigma\tau = I\frac{d^2\theta}{dt^2}$$

Where τ= torque of a force, and I is the moment of inertia. [5]
Similarly, torque is also equal to the moment arm times the component of force perpendicular to the moment arm. [6] Hence the final equation comes out to be:

$I\frac{d^2\theta}{dt^2}$= mgdsinθ (1)

Where mgsinθ is the vertical component of force to the moment arm. For the above equation torque- moment of inertia relation is used.
Using Heisenberg uncertainty principle at t=0,
$d.\theta_0.m.d.\omega_0 \geq \frac{\hbar}{2}$ (2)

For small angles, sinθ∼θ, so equation (1) can be written as
$\frac{d^2\theta}{dt^2} = \alpha^2\theta$ (3), where $\alpha = \sqrt{\frac{mgd}{I}}$

The solution for equation (3) is
$\theta(t) = Ae^{\alpha t} + Be^{-\alpha t}$ (4)

In equation (4), A and B are constants, depending on initial conditions. [7]

Using the value of θ at t=0 and using the value of ω at t=0
$A + B = \theta_0$ (5)
$\alpha(A - B) = \omega_0$ (6)

Solving equations (5) and (6), we get
$A= (\theta_0+ \omega_0/\alpha)/2$
$B= (\theta_0- \omega_0/\alpha)/2$

Using the above values in equation (4) we get,
$\theta(t) = \frac{1}{2}(\theta_0+ \omega_0/\alpha)e^{\alpha t} + \frac{1}{2}(\theta_0- \omega_0/\alpha)e^{-\alpha t}$ (7)

Since the constants A and B are small i.e. in the order of $\sqrt{h}$, therefore, by the time the positive exponential has increased enough to make θ of order 1, the negative exponential will have become negligible. So, we can ignore the part with coefficient B, and the equation (7) becomes:
$\theta(t) = \frac{1}{2}(\theta_0+ \omega_0/\alpha)e^{\alpha t}$ (8)

For finding the maximum time what the structure will be stable, θ should be small for as long as possible. Therefore, we have to minimize the coefficient of the exponential, subject to the uncertainty principle.

From eq(2), we have
$\omega_0 \geq \hbar/(2md^2\theta_0)$.

For limiting conditions, we have
$\omega_0 = \hbar/(2md^2\theta_0)$.

The coefficient of exponential in terms of θ_0, α , m, d is
$C = \frac{1}{2}\left(\theta_0 + \hbar/(2md^2\theta_0\alpha \right)$

Differentiating C with respect to θ_0 and equating to 0, we get the minimum value for θ_0, which is
$\theta_0 = \sqrt{\frac{\hbar}{2m\alpha d^2}}$

Plugging this value in equation (8), and simplifying we get,
$\theta(t) = \sqrt{\frac{\hbar}{2\alpha m d^2}} e^{\alpha t}$ (9)

Let us have $\theta \approx 1$ as the limit of angular displacement for which the rod seems stable for a normal observer. Setting $\theta \approx 1$, and then solving for t gives

$t = \frac{1}{2\alpha} \ln\left(\frac{2md^2\alpha}{\hbar}\right)$ (10)

Expanding equation (10) to include the value of α, we get

$2t = \sqrt{\frac{I}{mgd}} \mathrm{Ln}\left(\frac{2md^2\sqrt{mgd/I}}{\hbar}\right)$ (11)

For a rod standing on its tip, its moment of inertia is $ml^2/38$ and $d = l/2$. [8]
So, the expression for the maximum time a normal-sized rod can be balanced is
$t = \frac{1}{4}\sqrt{\frac{2l}{3g}} \mathrm{Ln}\left(\frac{3m^2l^3g}{8\hbar^2}\right)$

III.DERIVATION OF TIME EXPRESSION FOR EVEN N- SIDED POLYGON

As shown in figure 1, a polygon with an even number of sides is balanced on one of its vertices on a platform.

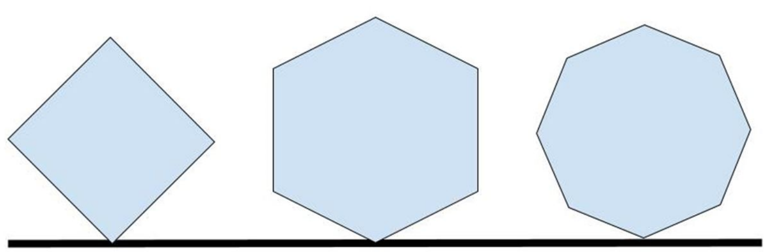

Fig.1 N-sided polygon standing on one of its vertices

To find time for maximum stability, the 2-d polygonal system can be simplified to a 1-d rod system, using a center of mass rotating about one of its vertices. So, To derive the required expression, a general equation for a distance of the center of mass from the point base, and moment of inertia needs to be calculated.

IV. CALCULATING THE POSITION OF CENTRE OF MASS AND MOMENT OF INERTIA

A. Distance of Base from the Center of Mass

Let a polygon with n number of sides be taken, where n= even number. Each side of the polygon subtends an angle of $2\pi/n$. As shown in the diagram below, using symmetry, we can construct a right-angled triangle.

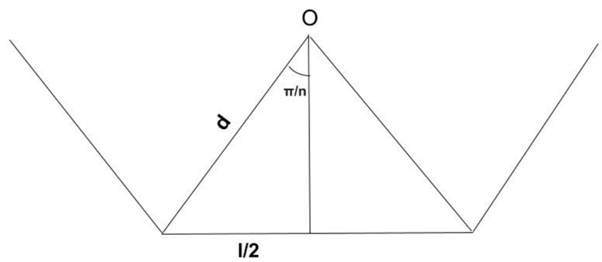

Fig. 2 A section of an n-sided polygon

Using trigonometry we get,

$$d = \frac{l}{2}\mathrm{cosec}(\pi/n) \qquad (12)$$

B. Moment Of Inertia Of A Regular Polygon About The Base

For a regular polygon made of sides of length l and of mass m. Since the polygon rotates about one of its vertices, the moment of inertia is to be found about that point. For this, moment of inertia about the center of mass is first needed. Then, the parallel axis theorem is applied to find the moment of inertia about a vertex.

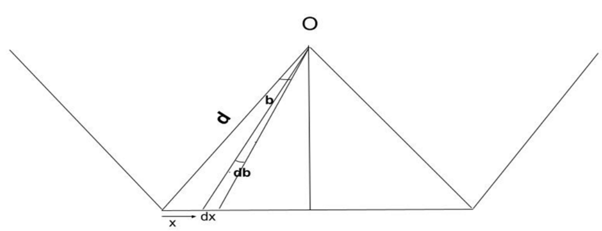

Fig. 3 Taking mass element for calculation of moment of inertia of polygon about its centre

To find the moment of inertia of polygon about the point O, we first find the moment of inertia of one side about the center of the polygon. For this, we take a small element dx at a distance x from the left vertex. Similarly, the angle subtended by x part of side be b, and the angle subtended by element dx be db.

We know

$I=\int_{r1}^{r2} dmr^2$, where r= distance of mass dm from point of interest. [9]

$$I= \int_{0}^{l/2} \frac{m}{l}dx.\,p^2 sec^2(\pi/n - b), \qquad (13)$$

where p= perpendicular distance from the center of the polygon to the side

Using trigonometry, the relation between x and angle b comes out to be

$1/p^2(l/2-x)^2 = 1-sec^2(\pi/n - b)$

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Using the above relation in equation (13), and simplifying we get the moment of inertia of half side to be

$$I = \frac{md^2}{l} \cdot \int_0^{l/2} \left(1 - \frac{1}{d^2}(l/2 - x)^2\right) dx$$

For the moment of inertia of the full side, the value from the above integral should be doubled which in turn should be multiplied by n to get the moment of inertia of the whole polygon.

Solving, the moment of inertia of the whole polygon about center comes out to be:

$$I = \frac{nml^2}{24}\left(1 + 3cot^2(\pi/n)\right) \qquad (14)$$

Using the parallel axis theorem for calculation of moment of inertia, the moment of inertia about vertex can be calculated:

$I = I_{com} + nmd^2$ [10]

Here equation (14) gives the value for I_{com} and equation (12) gives the value of d, we get,

$$I = \frac{1}{24}\left(nml^2(1 + 3cot^2(\pi/n) + 6cosec^2(\pi/n)\right) \qquad (15)$$

V. EXPRESSION FOR THE MAXIMUM TIME OF STABILITY

Using equation 11, we have

$$2t = \sqrt{\frac{I}{mgd}} \, Ln\left(\frac{2md^2\sqrt{mgd/I}}{\hbar}\right)$$

Putting the derived values for d, I, and m for general polygon case, we get:

$$t = \frac{1}{2}\left(\sqrt{\frac{\frac{1}{24}(nml^2(1 + 3cot2(\pi/n) + 6cosec2(\pi/n))}{nmg\frac{l}{2}cosec(\pi/n)}} \, Ln\left(\frac{2nm(\frac{l}{2}cosec(\pi/n))^2\sqrt{\frac{mg\frac{l}{2}cosec(\pi/n)}{\frac{1}{24}(nml^2(1 + 3cot2(\pi/n) + 6cosec2(\pi/n))}}}{\hbar}\right)\right)$$

Our goal is to examine the trend of time value with respect to the number of sides of the polygon. The graph for the above equation is plotted using graphing software, where time is on the y-axis and n is on the x-axis, as shown in the plot below:

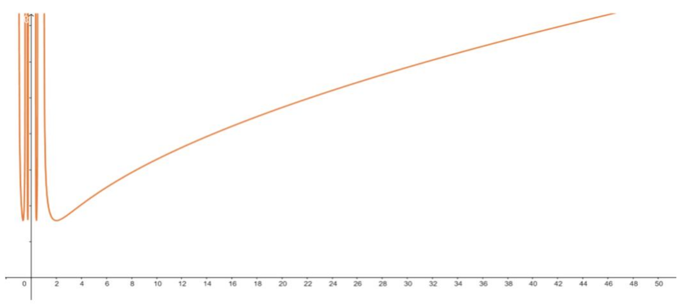

Fig. 4 A plot showing the relation between maximum stability time and number of sides of a polygon

As can be interpreted from the graph, as intended, the equation works for polygon only (i.e. n>2). The stability time goes on increasing as the number of sides of the polygon goes on increasing. For an even-n sided polygon, the stability is least for square (i.e. n=4), and highest for a circle(i.e. n tending to infinity)

VI. CONCLUSIONS

The stability equation for point base regular polygons is a marvelous example of how quantum mechanics creates visible effects in the macro world and plays a key role in the stability of different structures. It introduces the fact that among regular polygons standing on a point base, a circle has the highest stability time frame.

VII. ACKNOWLEDGMENT

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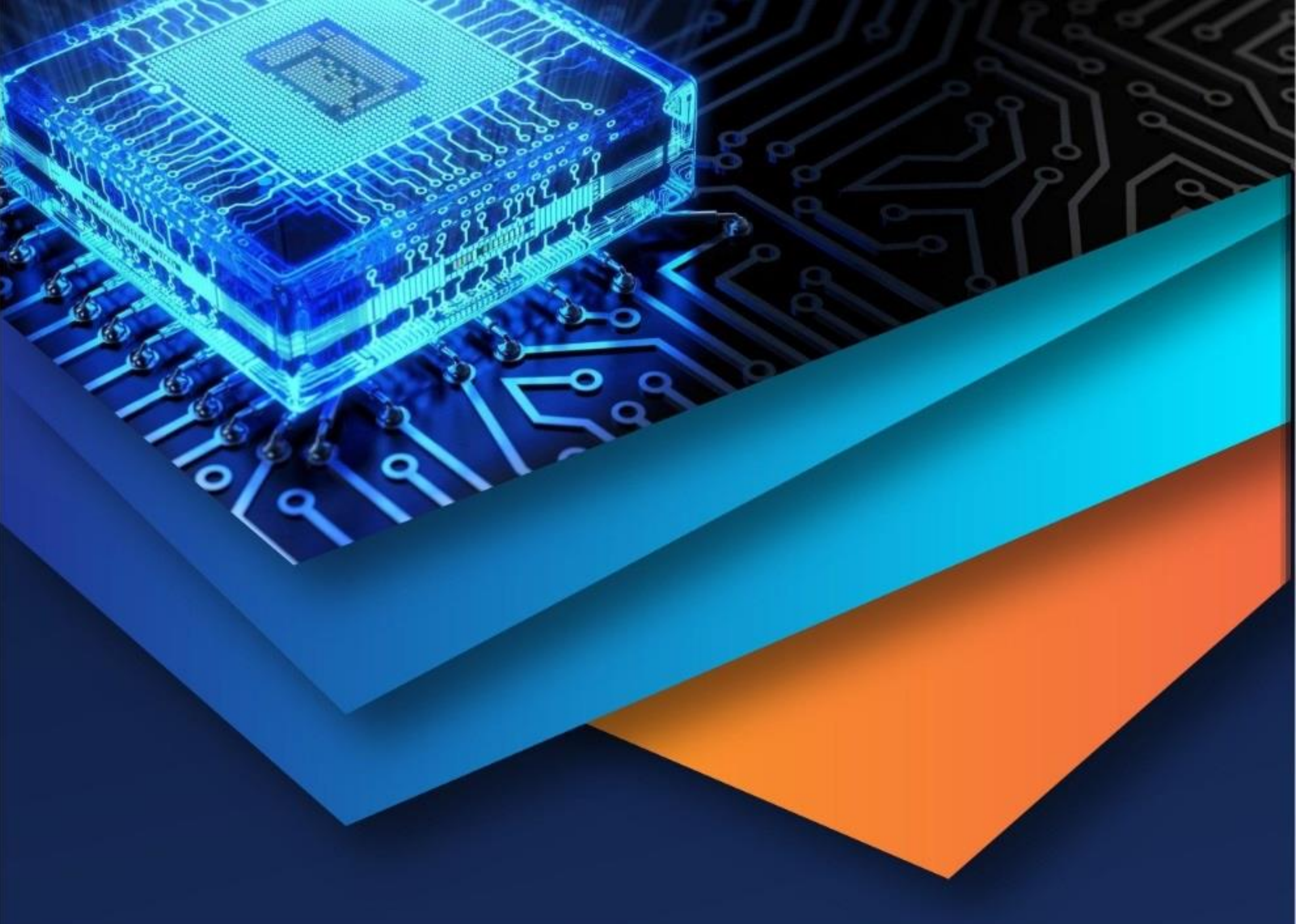

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