Volume 4 Issue XII, December 2016
ISSN: 2321-9653

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

Conquences of A Fixed Point Theorem for Quasi-Contractions of D*-Metric Spaces

Dr. C. Vijender
Department of Mathematics, Sreenidhi Institute of Science and Technology, Hyderabad

Abstract: The main aim of this paper is to prove the consequences of existence of fixed point on λ-generalized contraction of self mapping functions on D*-metric space.

Key Words: D*-metric space, Quasi-contraction, self mappings,

I. INTRODUCTION

The notion of Quasi-contraction defined for selfmaps of metric spaces given by Lj. B. Ciric [3] has been extended to the selfmaps of *D**-metric spaces as follows:

A. *Definition*

Let f be a selfmap of a *D**-metric space (*X*, *D**) and $x \in X$, $n \geq 1$ be an integer. The ***orbit of*** x ***under f of length n***, denoted by $O_f(x:n)$, is defined by

$$O_f(x:n) = \{x, fx, f^2x, \ldots, f^nx\}$$

We define the diameter $\delta(A)$ of a set A in a *D**-metric space (*X*, *D**) by $\delta(A) = \sup_{x,y \in A}\{D^*(x,y,y)\}$

The following Lemmas are use full in proving fixed point theorems of quasi-contractions on *D**-metric spaces:

B. *Definition*

A selfmap *f* of a *D**-metric space (*X*, *D**)is called *a* ***Quasi-contraction***, if there is a number q with $0 \leq q < 1$ such that

$$D^*(fx, fy, fy) \leq q.\max\{ D^*(x,y,y), D^*(x,fx,fx), D^*(y,fy,fy),$$

$$D^*(x,fy,fy), D^*(y,fx,fx) \}$$

for all $x, y \in X$.

As already noted that for every λ-generalized contraction is a quasi-contraction. However the following example gives a quasi-contraction f on a *D**-metric space (*X*, *D**) which is not a λ-generalized contraction.

II. PRELIMINARY NOTES

A. *Definition*

Let f be a selfmap of a *D**-metric space (*X*, *D**) and $x \in X$, $n \geq 1$ be an integer. The ***orbit of*** x ***under f of length n***, denoted by $O_f(x:n)$, is defined by

$$O_f(x:n) = \{x, fx, f^2x, \ldots, f^nx\}$$

We define the diameter $\delta(A)$ of a set A in a *D**-metric space (*X*, *D**) by $\delta(A) = \sup_{x,y \in A}\{D^*(x,y,y)\}$

Volume 4 Issue XII, December 2016
ISSN: 2321-9653
International Journal for Research in Applied Science & Engineering Technology (IJRASET)

The following Lemmas are use full in proving fixed point theorems of quasi-contractions on D^*-metric spaces:
In this section we prove

B. Lemma

Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and n be a positive integer. Then for each $x \in X$ and all integers $i, j \in \{1, 2, 3, \ldots, n\}$,

$$D^*\left(f^i x, f^j x, f^j x\right) \le q.\delta\left[O_f\left(x:n\right)\right] < \delta\left[O_f\left(x:n\right)\right].$$

Proof: Let $x \in X$ be arbitrary, $n \ge 1$ be an integer and $i, j \in \{1, 2, 3, \ldots, n\}$. Then $f^{i-1}x, f^{j-1}x, f^i x, f^j x \in O_f\left(x:n\right)$ and since f is a quasi-contraction,

$$D^*\left(f^i x, f^j x, f^j x\right) = D^*\left(ff^{i-1}x, ff^{j-1}x, ff^{j-1}x\right)$$

$$\le q.\max\left\{D^*\left(f^{i-1}x, f^{j-1}x, f^{j-1}x\right), D^*\left(f^{i-1}x, f^i x, f^i x\right),\right.$$
$$D^*\left(f^{j-1}x, f^j x, f^j x\right), D^*\left(f^{i-1}x, f^j x, f^j x\right),$$
$$\left.D^*\left(f^{j-1}x, f^i x, f^i x\right)\right\}$$

$$\le q.Sup\left\{D^*\left(u,v,v,\right): u,v \in O_f\left(x:n\right)\right\}$$

$$= q.\delta\left[O_f\left(x:n\right)\right]$$

$$< \delta\left[O_f\left(x:n\right)\right]$$

C. Lemma

Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and $x \in X$, then for every positive integer n, there exists positive integer $k \le n$, such that

$$D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f\left(x:n\right)\right]$$

Proof: If possible assume that the result is not true. This implies that there is positive integer m such that for all $k \le m$, we have $D^*\left(x, f^k x, f^k x\right) \ne \delta\left[O_f\left(x:m\right)\right]$. Since $O_f\left(x:m\right)$ contains x and $f^k x$ for $k \le m$, it follows that

$$D^*\left(x, f^k x, f^k x\right) < \delta\left[O_f\left(x:m\right)\right]$$

Since $O_f\left(x:m\right)$ is closed, there exists $i, j \in \{1, 2, 3, \ldots, m\}$ such that $D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f\left(x:m\right)\right]$, contradicting the Lemma 2.1. Therefore

$$D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f\left(x:n\right)\right] \text{ for some } k \le n.$$

D. Lemma

Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*), then

$$\delta\left[O_f\left(x:\infty\right)\right] \le \frac{1}{1-q} D^*\left(x, fx, fx\right) \text{ for all } x \in X.$$

Proof: Let $x \in X$ be arbitrary. Since $O_f(x:1) \subseteq O_f(x:2) \subseteq \ldots \subseteq O_f(x:n) \subseteq O_f(x:n+1) \subseteq \ldots$, we get that

$$\delta\left[O_f(x:1)\right] \le \delta\left[O_f(x:2)\right] \le \ldots \le \delta\left[O_f(x:n)\right] \le \delta\left[O_f(x:n+1)\right] \le \ldots,$$ showing

$$\lim_{n\to\infty} \delta\left[O_f(x:n)\right] = Sup\left\{\delta\left[O_f(x:n)\right] : n = 1, 2, 3, \ldots\right\}.$$

Therefore to prove the Lemma, it is enough to show

III. MAIN RESULT

A. Theorem

Suppose f is a selfmap of a D^*-metric space (X, D^*) and X is f-orbitally complete. If there is a positive integer k such that f^k is a quasi-contraction with constant q. Then f has a unique fixed point $u \in X$. In fact,

$$u = \lim_{n\to\infty} f^n x \text{ for any } x \in X$$

and

$$D^*\left(f^n x, u, u\right) \le \frac{q^n}{1-q} a(x) \text{ for all } x \in X,\ n \ge 1,$$

where $a(x) = \max\left\{D^*\left(f^i x, f^{i+k} x, f^{i+k} x\right) : i = 1, 2, 3, \ldots\right\}$ and $m = \left[\frac{n}{k}\right]$, the greatest integer not exceeding $\frac{n}{k}$.

Proof: Suppose f^k is a quasi-contraction of a D^*-metric space (X, D^*). It has unique fixed point by Theorem 3.1. Let u be a fixed point of f^k. Then we claim that fu is also a fixed point of f^k. In fact,

$$f^k(fu) = f^{k+1}u = f\left(f^k u\right) = fu$$

By the uniqueness of fixed point of f^k, it follows that $fu = u$, showing that u is a fixed point of f. Uniqueness of the fixed point of f can be proved as in the Theorem 3.1.

To prove (3.3), let n be any integer. Then by the division algorithm, we have, $n = mk + j,\ 0 \le j < k,\ m \ge 0$

Therefore $x \in X,\ f^n x = \left(f^k\right)^m f^j x,$ since f^k is a quasi-contraction,

$$D^*\left(f^n x, u, u\right) \le \frac{q^m}{1-q} D^*\left(f^j x, f^k f^j x, f^k f^j x\right)$$

$$\le \frac{q^m}{1-q} . \max\left\{D^*\left(f^i x, f^k f^i x, f^k f^i x\right) : i = 0, 1, 2, \ldots, k-1\right\}$$

$$\le \frac{q^m}{1-q} . \max\left\{D^*\left(f^i x, f^{k+i} x, f^{k+i} x\right) : i = 0, 1, 2, \ldots, k-1\right\}$$

proving (3.3). Letting $m \to \infty$, we get that $\lim_{n \to \infty} f^n x = u$, since $q^m \to 0$ as $m \to \infty$, proving (3.2). This completes the proof of the theorem.

B. Theorem

Let f be a quasi-contraction with constant q on a metric space (*X, d*) and X be *f*-orbitally complete, then f has a unique fixed point $u \in X$. In fact,

$$u = \lim_{n \to \infty} f^n x \text{ for all } x \in X$$

and

$$d\left(f^n x, u\right) \le \frac{q^n}{1-q} . d\left(x, fx\right) \text{ for all } x \in X,\ n \ge 1.$$

Proof: If (X, *d*) is a *f*-orbitally complete metric space, then it can be proved that $\left(X, D_1^*\right)$ is a *f*-orbitally complete *D**-metric space and hence *f*-orbitally complete for any selfmap f of *X*. Also if f is a quasi-contraction with constant q of (*X, d*), then the condition of quasi-contraction can be written as

$$D_1*\left(fx, fy, fy\right) \le q.\max\left\{D_1*\left(x, y, y\right), D_1*\left(x, fx, fx\right), D_1*\left(y, fy, fy\right), D_1*\left(x, fy, fy\right), D_1*\left(y, fx, fx\right)\right\}$$

for all $x, y \in X$, since $D_1^*\left(x, y, y\right) = d\left(x, y\right)$; so that f is a quasi-contraction on $\left(X, D_1^*\right)$. Thus f is a quasi-contraction on the *f*-orbitally complete *D**-metric space $\left(X, D_1^*\right)$ and hence the conclusions of Theorem 3.1 hold for f; which are the conclusions of the theorem.

REFERENCES

[1] Gerald Jungck, "Compatible mappings and common fixed points", Inetrnet J. Math., Math. Sci 9 (1986); 771-779
[2] R. Kannan, "On certain sets and fixed point theorems", Roum. Math. Pure Appl. 14 (1969), 51-54
[3] R. Kannan, "Some results on fixed points- II", American Math. Monthly, 76 (1969), p (406)
[4] Solomon Leader, "Fixed point for general contraction in metric spaces", Math. Japanica, 24 (1979), pp 17-24
[5] S. V. R. Naidu, K.P.R. Rao and N. Srinivasa Rao, "On the topology of D-metric spaces and generation of D-metric spaces", Int. J. Math., Math. Sci. 51 (2004), pp 2719-2740.
[6] S. V. R. Naidu, K.P.R. Rao and N. Srinivasa Rao, "On concepts of balls in D-metric spaces", Int. J. Math., Math. Sci. 1 (2005), pp 133-141.
[7] S. V. R. Naidu, K.P.R. Rao and N. Srinivasa Rao, "On convergent sequences and fixed point theorems in D-metric spaces", Int. J. Math., Math. Sci. 1 (2005), pp 1969-1988.
[8] B.E. Rhoades, "A fixed point theorem for generalized metric spaces", Int. J. Math. Sci. 1 (2005).
[9] B. Singh and R.K. Sharma, "Common fixed points via compatible maps in D-metric space", Radovi Matemeticki 11 (2002); pp 145-153.
[10] Shaban Sedghi, Nabi Shobe and Haiyun Zhou, "A common fixed point theorem in D*-metric spaces", Journal of fixed point theory and applications, Haindavi Publishing corporation.