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Integrability of Trigonometric Series with Coefficients Satisfying Certain Conditions

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Abstract: ***Let*** $1 \le P < \infty$ ***and*** $-1 < \alpha P < P-1$, ***suppose that*** $\{a_n\}$ ***is a sequence of ,Numbers such that*** $a_n \in A_j$ ***or*** $a_n \in A_{-j}$ ***and*** $\left\{\sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n)(a_n)^P\right\}^{1/P} < \infty$, ***then we will prove that*** $L^{1/P}\left(\frac{1}{x}\right) f(x) \in L(P,\alpha)$

And $\left\| L^{1/P}\left(\frac{1}{x}\right) f(x) \right\|_{P,\alpha}^{P} \le B(\alpha, P, j) \sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n)(a_n)^P$

& ii)Let $\{a_n\}$ ***be a sequence of numbers such that*** $a_n \in A_j$ ***or*** $a_n \in A_{-j}$. ***If*** $1 \le P < \infty$ ***and*** $-1 < \alpha P < P-1$, ***then a necessary and sufficient condition that*** $L^{1/P}\left(\frac{1}{x}\right) f(x) \in L(P,\alpha)$ ***is that*** $\sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n) a_n^{\,P} < \infty$.

I. INTRODUCTION

A. Definitions

A function $\phi(x)$ is said to belong to class $L(P,\alpha)$ if $\int_0^{\pi} |\phi(x)|^P (\sin x)^{\alpha P}\, dx < \infty$, α is a real number and P>0, it is easy to see that

$L(P,\alpha) \Rightarrow L^P$ *for* $\alpha < 0$ And

$L^P \Rightarrow L(P,\alpha)$ *for* $\alpha > 0$ And $L(P,\alpha) = L^P$ *if* $\alpha = 0$.

We define norm of a function $\phi(x) \in L(P,\alpha)$ as: $\|\phi(x)\|_{P,\alpha} = \left\{ \int_0^{\pi} |\phi(x)|^P (\sin x)^{\alpha P}\, dx \right\}^{1/P}$

A positive continuous function $L(x)$ is said to be "slowly increasing", in the sense of *Karamata [4]* if

$\lim_{x \to \infty} \frac{L(kx)}{L(x)} = 1$ For every $k > 0$.

A sequence $\{a_n\}$ of non-negative number is said to be quasi-monotone *[7, 9]* if for some constant $\alpha \ge 0$

$a_{n+1} \le a_n \left(1 + \frac{\alpha}{n}\right)$ for all $n > n_0(\alpha)$.

An equivalent definition is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$ .We shall say that the coefficients of trigonometric series,

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ and $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$ belong to the class A_j if for some $j \ge 0$, the number $n^{-j} a_n$, $a_n \ge 0$,

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decreases and to the class A_{-j} if, for some $j>0$, the number $n^j a_n$, $a_n \geq 0$ increases. The coefficients decrease monotonically to zero belongs to the class A_0.

B. Some known results

Theorem_ If $0<\upsilon<1, a_n \downarrow 0$, then $x^{-\upsilon} L\left(\frac{1}{x}\right) f(x) \in L(0,\pi)$ if and only if $\sum_{n=1}^{\infty} n^{\upsilon-1} L(n) a_n$ is convergent.

Theorem__If $a_n \downarrow 0$, $P \geq 1$, and $-1<\upsilon<0$, then the necessary and sufficient condition that $\sum_{n=1}^{\infty} n^{-1+P\upsilon+P} L(n) a_n{}^P$ should converge, is that $x^{-1-P\upsilon} L\left(\frac{1}{x}\right) f^P(x) \in L(0,\pi)$.

Theorem_ Let $\{a_n\}$ is quasi-monotone if $\alpha<1$ and such that $0<M_1 \leq n^{\beta} L_1(n) a_n \leq M_2$ with $\beta>0$, if $P \geq 1$ and $1-P<\lambda<1$. Then,

$f(\lambda, L, P)=x^{-\lambda} L_2\left(\frac{1}{x}\right) f^P(x)$ is integrable in $(0,\pi)$ if and only if $\sum_{n=1}^{\infty} n^{\lambda+P-2} L_2(n) a_n{}^P<\infty$, where L_1 and L_2 are slowly increasing function in the sense of *Karamata.*

Theorem__Let a_n be positive and tends to zero. Let $a_n n^{-k}$ be monotonically decreasing for some non-negative integer k. Let $1 \leq P<\infty$ and $-1<\alpha P<P-1$, then a necessary and sufficient condition that $f(x) \in L(P,\alpha)$, where $f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx$ is that

$$\sum_{n=1}^{\infty} n^{P-\alpha P-2} a_n{}^P<\infty .$$

Theorem_ Let $\{a_n\}$ be a positive sequence tending to zero and $\{a_n n^{-k}\}$ be monotonically decreasing for some non-negative integer k. If $1 \leq P<\infty$ and $-1<\alpha P<P-1$ then the necessary and sufficient condition that $L^{1/P}\left(\frac{1}{x}\right) f(x) \in L(P,\alpha)$ is that, $\sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n) a_n{}^P<\infty$, where $f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx$.

C. Our theorems

We shall prove the following theorems

Theorem_ Let $1 \leq P<\infty$ and $-1<\alpha P<P-1$, suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \in A_j$ or $a_n \in A_{-j}$ and $\left\{\sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n)(a_n)^P\right\}^{1/P}<\infty$,

Then, $L^{1/P}\left(\frac{1}{x}\right) f(x) \in L(P,\alpha)$

And $\left\|L^{1/P}\left(\frac{1}{x}\right) f(x)\right\|_{P,\alpha}^{P} \leq B(\alpha, P, j) \sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n)(a_n)^P$

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Theorem Let $\{a_n\}$ be a sequence of numbers such that $a_n \in A_j$ or $a_n \in A_{-j}$. If $1 \le P < \infty$ and $-1 < \alpha P < P-1$, then a necessary and sufficient condition that $L^{1/P}\left(\frac{1}{x}\right) f(x) \in L(P,\alpha)$ is that $\sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n) a_n^{\ P} < \infty$.

D. Lemmas

The following lemmas will be required for the proof of our theorems

1) *Lemma:* Let $f(x) \ge 0$ for $(x) \ge 0$ and f(x) be the integral of f(x). If $1 \le P < q$ and r>-1 then

$$\left\{\int_0^{\infty} t^{-1-qr}\left(L\left(\frac{1}{t}\right)\frac{f(t)}{t}\right)^q dt\right\}^{1/q} \le B\left\{\int_0^{\infty} t^{-1-\mathrm{Pr}}\left(L\left(\frac{1}{t}\right) f(t)\right)^P dt\right\}^{1/P}$$

$$\left\{\int_0^{\infty} t^{-1-qr}\left(L(t)\frac{f(t)}{t}\right)^q dt\right\}^{1/q} \le B\left\{\int_0^{\infty} t^{-1-\mathrm{Pr}}\left(L(t) f(t)\right)^P dt\right\}^{1/P}$$

2) *Lemm :* Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and $A_n = \sum_{k=n}^{\infty} a_k$ and suppose that

$\sum_{n=1}^{\infty} n^{-c} L(n)(na_n)^{p,} < \infty, (P \ge 1, c < 1)$, then $\sum_{n=1}^{\infty} n^{-c} L(n) A_n^{\ P} \le B \sum_{n=1}^{\infty} n^{-c} L(n)(na_n)^{p,}$, where B is some constant depending upon c and P.

3) *Lemma:* For any non-negative υ and n, we have

$$\left|\sum_{k=2^{\upsilon}(n+1)}^{2^{\upsilon+1}(n+1)-1} a_k \phi(kx)\right| \le \begin{cases} \dfrac{2j}{\left|\sin \frac{x}{2}\right|} a_{2^{\upsilon}(n+1),}, \mathit{if}\ \ a_k \in A_j \\ \dfrac{2j}{\left|\sin \frac{x}{2}\right|} a_{2^{\upsilon+1}(n+1)-1,}, \mathit{if}\ \ a_k \in A_{-j} \end{cases}$$

$x \ne 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots\ldots$ Where $\phi(x) = \cos x$ or $\phi(x) = \sin x$

4) *Lemma:* If $a_k \in A_j$ or $a_k \in A_{-j}$ then

$$\sum_{\upsilon=1}^{\infty}\left[2^{\upsilon}(n+1)\right]^{1+\alpha} a_{2^{\upsilon}(n+1)} \le c_1(\alpha, j)\sum_{k=n+1}^{\infty} k^{\alpha} a_k \ \text{ if } a_k \in A_j,$$

$$\sum_{\upsilon=0}^{\infty}\left[2^{\upsilon}(n+1)\right]^{1+\alpha} a_{2^{\upsilon}(n+1)} \le c_2(\alpha, j)\sum_{k=n+1}^{\infty} k^{\alpha} a_k \ \text{ if } a_k \in A_{-j},$$

Where,

$$C_1(\alpha, j) = \begin{cases} 2 & \mathit{for}\ \alpha + j \le 0 \\ 2^{1+\alpha+j} & \mathit{for}\ \alpha + j > 0 \end{cases}$$

$$C_2(\alpha, j) = \begin{cases} 1 & \mathit{for}\ \alpha - j \ge 0 \\ 2^{j-1} & \mathit{for}\ \alpha - j < 0 \end{cases}$$

5) *Lemma:* Let $\{a_n\}$ be a sequence of non-negative terms and

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$A(n)=\sum_{\left(\frac{n}{2}\right)+1}^{n}, \quad A^{*}(n)=\sum_{n}^{2n} a_k$ Then

$na_n \le B(j)A(n) \qquad if \ a_k \in A_j$

$na_n \le B(j)A^{*}(n) \qquad if \ a_k \in A_{-j}$

Where $B(j)$ is some positive constant depending on j.

Proof:

$$\sum_{k=s}^{m} \frac{a_k}{k^j} K^j \ge \frac{a_m}{m^j} s^j (m-s+1), \qquad if \quad a_k \in A_j,$$

$$\sum_{k=s}^{m} a_k K^j k^{-j} \ge a_s s^j m^j (m-s+1), \quad if \quad a_k \in A_{-j},$$

We set $s=\left[\frac{n}{2}\right]+1, m=n$ in (i) and $s=n$, $m=2n$ in (ii), we have

$$na_n \le \frac{A_n}{\left[\left[\frac{n}{2}\right]+1\right]^j} n^{j+1} \le B(j)A(n)$$

And $na_n \le B(j)A*(n)$.

This completes the proof of the lemma.

E. Proof of Theorem

Since we are given that $\sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n) a_n^{\ P} < \infty$, it follows by virtue of *Lemma2* (putting $c=-p+\alpha p+2$ and $a_k = {}^{a_k}\!/_{k}$) that $\sum_{n=1}^{\infty} n^{P-\alpha P-2} L(n) \left(\sum_{k=n}^{\infty} \frac{a_k}{k} \right)^p < \infty$, further on putting n=1, we have $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$.

Put $R_n(x) = \left| \sum_{k=n+1}^{\infty} a_k \cos kx \right| \le \left| \sum_{\upsilon=0}^{\infty} \quad \sum_{k=2^{\upsilon}(n+1)}^{2^{\upsilon+1}(n+1)-1} a_k \cos kx \right|$

$$\le \frac{2j}{\left|\sin \frac{x}{2}\right|} \begin{cases} \sum_{\upsilon=0}^{\infty} a_{2^{\upsilon}(n+1),}, & if \ a_k \in A_j \\ \sum_{\upsilon=0}^{\infty} a_{2^{\upsilon+1}(n+1),}, & if \ a_k \in A_{-j} \end{cases}$$

$x \ne 2k\pi,\ k=0,\pm1,\pm2,........$ by virtue of *lemma3* (choosing $\phi(x)=\cos x$)

Now using the particular case, when $\alpha=-1$ of *lemma 4,* we obtain

$$R_n(x) \le \frac{B^j}{\left|\sin \frac{x}{2}\right|} \left[a_{n+1} + \sum_{k=n+1}^{\infty} \frac{a_k}{k} \right], x \ne 2k\pi$$

Therefore series $\sum_{n=1}^{\infty} a_n \cos nx$ converges uniformly and is a Fourier series of the function *f(x)* which is continuous in $(0,2\pi)$. We have $\left|f(x)\right| = \left|\sum_{k=1}^{\infty} a_k \cos kx\right| = \left|\sum_{k=1}^{n} a_k \cos kx + \sum_{k=n+1}^{\infty} a_k \cos kx\right|$

$$\le \sum_{k=1}^{n} a_k + \frac{B^j}{x}\left[a_{n+1} + \sum_{k=n+1}^{\infty} \frac{a_k}{k}\right]$$

$$\le \sum_{k=1}^{n} a_k + O\left(\frac{1}{x}\right)a_{n+1} + O\left(\frac{1}{x}\right)\sum_{k=n+1}^{\infty} \frac{a_k}{k}$$

$$\le s_n + O\left(\frac{1}{x}\right)a_{n+1} + O\left(\frac{1}{x}\right)\left(\sum_{k=n+1}^{\infty} \frac{a_k}{k}\right)$$

Where $s_n = \sum_{k=1}^{n} a_k$

Now $\int_0^{\pi/2} L\left(\frac{1}{x}\right)\left|f(x)\right|^p (\sin x)^{\alpha p}\, dx = \sum_{n=2}^{\infty} \int_{\pi/(n+1)}^{\pi/n} L\left(\frac{1}{x}\right)\left|f(x)\right|^p (\sin x)^{\alpha p}\, dx$

$$\le \sum_{n=2}^{\infty} \int_{\pi/(n+1)}^{\pi/n} L\left(\frac{1}{x}\right)\left\{S_n + O\left(\frac{1}{x}\right)a_{n+1} + O\left(\frac{1}{x}\right)\left(\sum_{k=n+1}^{\infty} \frac{a_k}{k}\right)\right\}^p (\sin x)^{\alpha p}\, dx$$

$$\le B(p)\sum_{n=2}^{\infty} \int_{\pi/(n+1)}^{\pi/n} L\left(\frac{1}{x}\right) S_n{}^p (\sin x)^{\alpha p}\, dx + B(p, j)\sum_{n=2}^{\infty} \int_{\pi/(n+1)}^{\pi/n} L\left(\frac{1}{x}\right)(a_{n+1})^p (\sin x)^{\alpha p - p}\, dx$$

$$+ B(p, j)\sum_{n=2}^{\infty} \int_{\pi/(n+1)}^{\pi/n} L\left(\frac{1}{x}\right)\left(\sum_{k=n+1}^{\infty} \frac{a_k}{k}\right)^p (\sin x)^{\alpha p - p}\, dx$$

$$\le B(\alpha, p)\sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) S_n{}^p + B(\alpha, p, j)\sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n)(a_{n+1})^p$$

$$+ B(\alpha, p, j)\sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n)\left(\sum_{k=n+1}^{\infty} \frac{a_k}{k}\right)^p$$

$= j_1 + j_2 + j_3$ (say)

Now put $a_{(x)} = a_n \;\; for\; n-1 \le x < n \qquad (n = 1,2,\ldots\ldots)$

And $A(x) = \int_0^{\pi} a(t)dt$ then, we have,

$$j_1 \le B(\alpha, p)\sum_{n=1}^{\infty} \int_n^{n+1} x^{-2-\alpha p} L(x) A^p(x) dx = B(\alpha, p)\int_1^{\infty} x^{-2-\alpha p + p}\left\{\frac{L(x)^{1/p} A(x)}{x}\right\}^p dx$$

On applying *lemma (ii)* (taking $q = p$ and $-1 - qr = p - \alpha p - 2$) we get,

$$j_1 \le B(\alpha, p)\int_1^{\infty} x^{-2-\alpha p + p} L(x)(a(x))^p\, dx$$

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$$= B(\alpha, p)\sum_{n=2}^{\infty}\int_{n-1}^{n} x^{-2-\alpha p+p} L(x)(a(x))^{p}(x)dx$$

$$\leq B(\alpha, p)\sum_{n=1}^{\infty} n^{p-2-\alpha p} L(n)(a_n)^{p}$$

$<\infty$, by hypothesis of
heses.
By virtue of lemma 2 the theorem,
$j_2 = O(1)$ by the hypotand by the hypothesis,
$j_3 = O(1)$
A similar method may be used to estimate

$$\int_{\pi/2}^{\pi} |f(x)|^{P} (\sin x)^{\alpha P} dx$$

This finishes proof of theorem.

F. Proof of Theorem

<u>Necessity</u>: Suppose that $L^{1/P}\left(\frac{1}{x}\right) f(x) \in L(P,\alpha)$, then we have to prove, $\sum_{n=1}^{\infty} n^{p-2-\alpha p} L(n)(a_n)^{p} < \infty$

Let $f_1(x) = \int_0^x f(u)du$, $f_2(x) = \int_0^x f_1(u)du$

Then $f_2(x) = \int_0^{\infty}\left(\sum_{k=1}^{\infty}\frac{a_k}{k}\sin ku\right)du$

$$= \sum_{k=1}^{\infty}\frac{a_k}{k}\int_0^x \sin ku \; du$$

$$= \sum_{k=1}^{\infty} a_k (1-\cos kx)k^{-2}$$

$$\geq \sum_{k=s}^{m} a_k (1-\cos kx)k^{-2} \qquad \text{(A)}$$

For any positive integers s *and m,*

<u>Case I</u>: When $a_k \in A_j$

Now set $x = \left[\frac{n}{2}\right]+1$ and $m = n$ and using the inequality $1-\cos nx \geq A\frac{nx^2}{2}$ for $\frac{\pi}{4(n+1)} \leq x \leq \frac{\pi}{4n}$, we have $A_n \leq Bn^2 f_2(x)$, where B is some constant. By virtue of *Lemma 5(i)*, it follows

$$na_n \leq B(j)A_n \leq B(j)n^2 f_2(x)$$

Now, $\sum_{n=1}^{\infty} n^{-2-\alpha p} L(n)(na_n)^{p}$

$$\leq \mathbf{B}(j)\sum_{n=1}^{\infty} n^{2p-2-\alpha p} L(n) \min(f_2(x))^p \qquad \left(\frac{\pi}{4(n+1)} \leq x \leq \frac{\pi}{4n}\right)$$

$$\leq B(j)\sum_{n=1}^{\infty} \int_{\frac{\pi}{4(n+1)}}^{\frac{\pi}{4n}} (\sin x)^{-2p+\alpha p} L(\tfrac{1}{x}) f_2{}^p(x)dx$$

$$\leq B(j)\int_0^{\frac{\pi}{4}} (\sin x)^{-p+\alpha p}\left\{\frac{L(\frac{1}{x})^{1/p} f_2(x)}{x}\right\}^p dx,$$

$$\leq B(\alpha,p,j)\int_0^{\frac{\pi}{4}} (\sin x)^{-p+\alpha p}\left\{(L(\tfrac{1}{x}))^{1/p}\left|f_1(x)\right|\right\}^p dx$$

$$\leq B(\alpha,p,j)\int_0^{\frac{\pi}{4}} (\sin x)^{\alpha p}\left\{L(\tfrac{1}{x})^{1/p}\left|f(x)\right|\right\}^p dx$$

$$= B(\alpha,p,j)\left\|L^{1/p}\left(\tfrac{1}{x}\right)f(x)\right\|_{p,\alpha}^p < \infty$$

This follows by *lemma1 (i)* ($q = p$, $\alpha p - p = -1 - pr$ and $\alpha p = -1 - pr$ respectively)

<u>Case II</u>: When $a_k \in A_j$, we set $s = n$ and $m = 2n$ in (A) and obtain

$$A_n^* \leq Bn^2 f_2(x) \qquad \text{For } \frac{\pi}{8(n+1)} \leq x \leq \frac{\pi}{8n}$$

By *lemma 5(ii)* we get $na_n \leq B(j)n^2 f_2(x)$

Now, $\sum_{n=1}^{\infty} n^{-2-\alpha p} L(n)(na_n)^p$

$$\leq \mathbf{B}(j)\sum_{n=1}^{\infty} n^{2p-2-\alpha p} L(n) \min(f_2(x))^p, \qquad \frac{\pi}{8(n+1)} \leq x \leq \frac{\pi}{8n}$$

$$\leq B(j)\sum_{n=1}^{\infty} \int_{\frac{\pi}{8(n+1)}}^{\frac{\pi}{8n}} (\sin x)^{-2p+\alpha p} L(\tfrac{1}{x}) f_2{}^p(x)dx$$

$\leq B(j)\int_0^{\frac{\pi}{8}} (\sin x)^{-p+\alpha p}\left\{\frac{L(\frac{1}{x})^{1/p} f_2(x)}{x}\right\}^p dx < \infty$ By some agreement as in the *case I* this possesses the necessity part of *theorem2.*

<u>*Sufficiency*</u>*:* Now suppose that $\sum_{n=1}^{\infty} n^{p-2-\alpha p} L(n)(na_n)^p \leq \infty$. Then we have to show $L^{1/P}\left(\frac{1}{x}\right) f(x) \in L(P,\alpha)$.

This follows by *theorem 1* and proof of the theorem is thus completed.

II. CONCLUSION

Theorem1 and theorem 2 also hold for sine series. The proof of sufficiency part for sine series follows exactly in a same way as in

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case of theorem1 while, for proof of necessity part, some miner changes are required.

For sake of convenience the *theorem 1* is stated and proved otherwise theorem is essentially the same as $\sum - \int$ part of theorem 2.

Our theorem 2 is not only more general then a result of *Askey and Wainger [2]* and theorem of *Khan [5],* but has a proof applicable in sine and cosine series both.

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