\(\hat{\delta}\omega\) Closed Sets in Ideal Topological Space

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Abstract: In this paper the notion of \(\hat{\delta}\omega\) closed sets is introduced and some of its basic properties are studied. This new class of sets is independent of semi closed and closed sets. Also the relationship with some of the known closed sets is discussed.

Keywords: \(\hat{\delta}\omega\) closed sets, closed sets, \(\omega\) closed sets.

I. INTRODUCTION

Levine, velicko introduced the notions of generalized closed (briefly gclosed) and \(\delta\) closed sets respectively and studied their basic properties. The notion of \(I_g\) closed sets was first introduced by Dontchev in 1999. Navaneetha Krishnan and Joseph further investigated and characterized \(I_g\) closed sets. Julian Dontchev and maximilian Ganster, Yuskel, Acikgoz and Noiri introduced and studied the notions of \(\delta\) generalized closed (briefly \(\delta\) g closed) and \(\delta\)-I-closed sets respectively. The purpose of this paper is to define a new class of sets called \(\hat{\delta}\omega\) closed sets and also study some basic properties and characterizations.

Throughout this paper \((X, \tau, I)\) represents an ideal topological space on which no separation axiom is assumed unless otherwise mentioned. For a subset \(A\) of a ideal topological space \(X\), \(\text{cl}(A)\) and \(\text{int}(A)\) denote the closure of \(A\) and the interior of \(A\) respectively. \(X / A\) or \(\complement X\) denotes the complement of \(A\) in \(X\). We recall the following definitions and results.

II. PRELIMINARIES

A. subset \(A\) of a space \(X\) is called

pre-open set if \(A \subseteq \text{intcl}(A)\) and pre-closed set if \(\text{clint}(A) \subseteq A\).

semi-open set if \(A \subseteq \text{clint}(A)\) and semi-closed set if \(\text{intcl}(A) \subseteq A\).

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regular open set if \(A = \text{intcl}(A)\) and regular closed set if \(A = \text{clint}(A)\).

\(I\) -open set if \(A\) is a finite union of regular open sets.

regular semi open if there is a regular open \(U\) such \(U \subseteq A \subseteq \text{cl}(U)\)

B. A subset \(A\) of \((X, \tau, I)\) is called

generalized closed set, if \(\text{cl}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is open in \(X\).

regular generalized closed set, if \(\text{cl}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).

weakly generalized closed set, if \(\text{clint}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is open in \(X\).

weakly closed set, if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi open in \(X\).

regular weakly generalized closed set, if \(\text{clint}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).

regular weakly closed if \(\text{cl}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is regular semi open.

g-closed if \(\text{cl}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is w-open.

Let \(A\) and \(B\) be subsets of an ideal topological space \((X, \tau, I)\). Then, the following properties holds.

\(A \subseteq \sigma\text{cl}(A)\).

If \(A \subseteq B\), then \(\sigma\text{cl}(A) \subseteq \sigma\text{cl}(B)\).

\(\sigma\text{cl}(A) = \bigcap \{F \subseteq X \mid A \subseteq F\text{ and }F\text{ is }\delta-I\text{-closed}\}\).

If \(A\) is \(\delta\)-I-closed set of \(X\) for each \(\alpha \in \Delta\), then \(\bigcap\{A\alpha/\alpha \in \Delta\}\) is \(\delta\)-I-closed.

\(\sigma\text{cl}(A)\) is \(\delta\)-I-closed.

\(\delta\)-I closure is \(\{x \in X : \text{int}(\text{cl}^*(U)) \cap A \neq \emptyset, U \in I\}\).
III. $\delta_\omega$ - CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

Definition 3.1: A subset $A$ of an ideal space $(X, \tau, I)$ is called $\delta_\omega$-closed, if $\sigma cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\omega$-open.

Theorem 3.1

Every g-closed set in $X$ is $\delta_\omega$-closed set in $X$.

Proof: Let $A$ be an arbitrary g-closed set in the space $X$. Suppose $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open. Then by the definition of $\delta_\omega$-closed set, if $\sigma cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\omega$-open in $X$. Hence, the arbitrary element $A$ of g-closed set belongs to $U$ and also the arbitrary element $A$ of $\delta_\omega$-closed set belongs to $U$. This implies $A$ is a $\delta_\omega$-closed set.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.1: Let $X = \{b, c, d\}$ be with topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$. Now if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$. Then g-closed set will be $\{\phi, \{b\}, \{b, c\}, \{b, d\}, X\}$. Here, $A = \{c\}$ is a set, $\delta_\omega$-closed set but not g-closed.

Theorem 3.2

Every closed set in $X$ is $\delta_\omega$-closed set in $X$.

Proof: Let $A$ be an arbitrary closed set in the space $X$, every closed set is g-closed set and from the theorem 3.1, every g-closed set in $X$ is $\delta_\omega$-closed. Thus every closed set in $X$ is $\delta_\omega$-closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.2: Let $X = \{b, c, d\}$ be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Thus, the closed set is $\{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Here $A = \{c\}$ is $\delta_\omega$-closed set but not closed set.

Theorem 3.3

Every regular closed set in $X$ is $\delta_\omega$-closed.

Proof: Let $A$ be an arbitrary regular closed set in the space $X$, every regular closed set is closed and from the theorem 3.1, every g-closed set in $X$ is $\delta_\omega$-closed. This implies every regular closed set in $X$ is $\delta_\omega$-closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.3: Let $X = \{b, c, d\}$ be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$, the regular closed set, here $A = \{c\}$ is $\delta_\omega$-closed set but not regular closed set.

Theorem 3.4

Every regular generalized closed set in $X$ is $\delta_\omega$-closed.

Proof: Let $A$ be an arbitrary regular generalized closed set in the space $X$. Suppose $cl(A) \subseteq U$. Whenever $A \subseteq U$ and $U$ is regular open, i.e., $A \subseteq U$ and $U$ is regular open, every regular open set in $X$ is open. Then by the definition of $\delta_\omega$-closed set, if $\sigma cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$. Hence, the arbitrary element $A$ of regular generalized closed set belongs to $U$ and the element $A$ of $\delta_\omega$-closed set belongs to $U$. This implies that $A$ is a $\delta_\omega$-closed set.

The converse of above theorem need not be true, which is verified from the following example.

Example 3.4: Let $X = \{b, c, d\}$ be with topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Then if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is regular open in $X$. Then regular generalized closed set will be $\{\phi, \{b, d\}, \{b, c\}, \{c, d\}, \{b\}, X\}$. Here $A = \{c\}$ is a $\delta_\omega$-closed set but not regular generalized closed set.

Theorem 3.5

Every weakly generalized closed set in $X$ is $\delta_\omega$-closed.

Proof: Let $A$ be an arbitrary weakly generalized closed set in the space $X$. Then by definition of weakly generalized closed set and $\delta_\omega$-closed set the arbitrary element $A$ of weakly generalized closed set belongs to $U$ and the arbitrary element $A$ of $\delta_\omega$-closed set belongs to $U$. This implies that $A$ is a $\delta_\omega$-closed set.

The converse of above theorem need not to be true, which is verified from the following example.

Example 3.5: Let $X = \{b, c, d\}$ be with topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$. Now, if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$. Then weakly-closed set will be $\{\phi, \{b, c\}, \{b, d\}, \{b\}, X\}$. Here $A = \{c\}$ is a $\delta_\omega$-closed set but not weakly generalized closed set.

Theorem 3.6

Every semi closed set in $X$ is $\delta_\omega$-closed.
Every weakly closed set in $\omega$ –closed set.

Proof: Let A be an arbitrary weakly closed set in the space X. Suppose cl(A) belongs to U whenever A belongs to U and U is semi open in X. i.e., whenever A belongs to U and U is open in X, every weakly open set is open in X. Then by the definition of $\omega$ –closed set, if cl(A) belongs to U and U is open in X. Hence, the arbitrary element A of weakly closed set belongs to U and the arbitrary element A of $\omega$ –closed set belongs to U. This implies that A is a $\omega$ –closed set in X.

The converse of the above theorem need not to be true, which is verified from the following example.

Example 3.7: Let X = {b, c, d} be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$. Now if cl(A) belongs to U whenever A belongs to U and U is semi open in X. Then $U = \{\phi, \{c\}, \{d\}, X\}$. Then weakly closed set will be $\{\phi, \{b\}, \{b, d\}, \{b, c\}, \{c, d\}, X\}$.

Here A = {c} is a closed set in $\omega$ - closed set, but not Weakly X.

Theorem 3.8

Every weakly closed set in X is $\omega$ - closed.

Proof: Let A be a regular weakly generalized closed set in the space X. Suppose cl(int(A)) belongs to U whenever A belongs to U and U is open in X. i.e., A and U are open in X, every regular open set in X is open. Then by the definition of $\omega$ closed set, if cl(A) belongs to U, whenever A belongs to U and U is $\omega$ open in X. Hence, the arbitrary element A of regular weakly generalized closed set belongs to U and the arbitrary element A of $\omega$ –closed set belongs to U. This implies that A is a $\omega$ –closed set in X.

Example 3.8: Let X = {b, c, d} be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$. Now if cl(int(A)) belongs to U whenever A belongs to U and U is regular open in X. Then $U = \{\phi, \{c\}, \{d\}, X\}$. Then regular weakly generalized closed set will be $\{\phi, \{b\}, \{b, d\}, \{b, c\}, \{c, d\}, X\}$. Here A = {c} is a set, $\omega$ closed set but not regular weakly generalized.

Theorem 3.9

Every regular semi closed set in X is $\omega$ - closed.

Proof: Let A be an arbitrary regular semi closed set in the space X. Suppose U belongs to A belongs to cl(U) whenever U is regular open set in X is open. i.e., U is open. Then by the definition of $\omega$ closed set, if cl(A) belongs to U, whenever A belongs to U and U is $\omega$ open in X. Hence, the arbitrary element A of regular semi closed set belongs to U and the arbitrary element A of $\omega$ –closed set belongs to U. Thus we can say that This implies that A is a $\omega$ –closed set.

The converse of the above theorem need not to be true, which is verified from the following example.

Example 3.9: Let X = {b, c, d} be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Then, from the definition of regular semi closed set is $\{\phi, \{c\}, \{d\}, \{b, d\}, \{b, c\}, \{c, d\}, X\}$. Here let A = {b} is $\omega$ –closed set but not regular semi closed.

Theorem 3.10

Every regular weakly closed set in X is $\omega$ - closed.

Proof: Let A be an arbitrary regular weakly closed set in the space X, every semi open set is open and from the theorem 3.2, every closed set in X is $\omega$ - closed set. This implies that every regular weakly closed set in X is $\omega$ - closed set.

The converse of the above theorem is not true, which is verified using following example $\omega$ closed sets in ideal topological space

Example 3.10: Let X = {b, c, d} be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$. Then by the definition of regular weakly closed set $\{\phi, \{c\}, \{d\}, \{b\}, \{b, c\}, \{b, d\}, X\}$. Here A = {c, d} is $\omega$ –closed set but not regular weakly closed.

Theorem 3.11

Every *g -closed set in X is $\omega$ - closed.

Proof: Let A be an arbitrary *g -closed set in the space X. Suppose cl(A) belongs to U whenever A belongs to U and U is semi open X. i.e., whenever A belongs to U and U is semi open, every weakly open set is open in X. Then by the definition of $\omega$ –closed set, if cl(A) belongs to U whenever A belongs to U and U is semi open in X. Hence, the arbitrary element A of weakly closed set belongs to U and the arbitrary element A of $\omega$ –closed set belongs to U. This implies that A is a $\omega$ –closed set in X.
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U, whenever A ⊆ U and U is ω open in X. Hence, the arbitrary element A of *g-closed set belongs to U and the arbitrary element A of δω-closed set belongs to U. This implies that A of δω-closed set in X.

The converse of the above theorem need not to be true, which is verified using following example.

Example 3.11: Let X = {b, c, d} be with the topology r = {∅, {c}, {d}, {c, d}, X}. Now if cl(A) ⊆ U, whenever A ⊆ U and U is weakly open in X. Then U = {∅, {c}, {d}, {c, d}, X}. Then *g-closed set will be {∅, {b}, {b, d}, {b, c}, X}. Here A = {c} is a δω-closed set, but not *g-closed set in X.

Theorem 3.12
Every ω-closed set in X is δω-closed set.

Proof: Let A be an arbitrary ω-closed set in space X. Suppose clω(A) ⊆ U, whenever A ⊆ U and U is open. i.e., A ⊆ U and U is open. Then by the definition of δω-closed set, if cl(A) ⊆ U, whenever A ⊆ U and U is ω-open in X. Hence, the arbitrary element A of ω-closed set belongs to U and also the arbitrary element A of δω-closed set belongs to U. This implies that, A is δω-closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.12: Let X = {b, c, d} be with the topology r = {∅, {c}, {d}, {c, d}}, X}. Now if clθ(A) ⊆ U, whenever A ⊆ U and U is open in X. Then ω-closed set will be {∅, {b}, {b, c}, {b, d}, {b}}. Here A = {c} is a δω-closed set, but not ω-closed set.

Theorem 3.13
Every δ-closed set in X is δω-closed set in X.

Proof: Let A be an arbitrary δ-closed set in space X. Suppose clδ(A) ⊆ U, whenever A ⊆ U and U is open. i.e., A ⊆ U and U is open. Then by the definition of δω-closed set, if cl(A) ⊆ U, whenever A ⊆ U and U is ω-open in X. Hence, the arbitrary element A of δ-closed set belongs to U and also the arbitrary element A of δω-closed set belongs to U. This implies that, A is δω-closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.13: Let X = {b, c, d} be with the topology r = {∅, {c}, {d}, {c, d}, X}. Now if clδ(A) ⊆ U, whenever A ⊆ U and U is open in X. Then δ-closed set will be {∅, {b}, {b, c}, {b, d}, {b}}. Here A = {c} is a δω-closed set, but not δ-closed.

IV. SOME OPERATIONS ON δω-CLOSED SETS

Theorem 4.1
The union of two δω-closed sets of X is also an δω-closed sets of X.

Proof: Assume that A and B are δω-closed sets in X. Let U be open in X, such that A ∪ B ⊆ U. Thus A ⊆ U and B ⊆ U. Since A and B are δω-closed sets so σcl(A) ⊆ U and σcl(B) ⊆ U. Hence σcl(A ∪ B) = σcl(A) ∪ σcl(B) ⊆ U. i.e., σcl(A ∪ B) ⊆ U Hence A ∪ B is an δω-closed set in X.

Theorem 4.2
If a subset A of X is δω-closed in X, then σcl(A)\A, A does not contain any non-empty open set in X.

Proof: Suppose that A is a δω-closed set in X. Let U be open set such that σcl(A)\A ⊆ U and U ≠ ∅. Now U ⊆ σcl(A)\A, i.e., U ⊆ X\A which implies that A ⊆ X\U. As U is open, X\U is also open in X. Since A is an δω-closed set in ideal topological space, A is closed in X. Therefore U ∩ (X\σcl(A)) = ∅ This show that U = ∅, which is contradiction. Hence σcl(A)\A does not contain any non-empty open set in X.

Theorem 4.3
For an element x ∈ X, the set X\{x} is δω-closed or ω-open.

Proof: Let x ∈ X. Suppose X\{x} is not ω-open. Then X is the only ω-open set containing X\{x}, which means that the only choice of an ω-open set containing X\{x} is X. i.e., X\{x} ⊆ X. As X is open, X\{x} is also open in X. Since cl(X\{x}) is not open. As cl(X\{x}) is a subset of X and X\{x} only but X\{x} is not δω-closed. Thus the only open set in X. Also σcl(X\{x}) ⊆ X. Therefore by the definition of δomega-closed sets X\{x} is δω-closed, which is a contradiction. Hence X\{x} is ω-open.

Theorem 4.4
Example 4.2: Let \( X = \{b, c, d\} \) be with the topology \( \tau = \{\emptyset, \{c\}, \{d\}, \{c, d\}, X\} \). 

The converse is not true, which is verified by following example.

If a subset \( A \) of a topological space \( X \) is \( \sigma \)-closed, we have \( \sigma(A) \subseteq \sigma(A) \) such that \( B \subseteq U \), where \( U \) is an \( \omega \)-open set. Thus \( \sigma(A) \subseteq U \), whenever \( B \subseteq \sigma(A) \) and \( U \) is \( \omega \)-open. Therefore \( B \) is an \( \delta \omega \)-closed set in \( X \).

Theorem 4.5

If a subset \( A \) of a topological space \( X \) is both \( \omega \)-open and \( \delta \omega \)-closed then it is \( \omega \)-closed.

Proof: Suppose a subset \( A \) of a topological space \( X \) is both \( \omega \)-open and \( \delta \omega \)-closed. Now \( A \subseteq A \) then by definition of \( \delta \omega \)-closed we have \( \sigma(A) \subseteq A \). Thus we have \( \sigma(A) = \sigma(A) \). Finally \( A \) is open.

Theorem 4.6

If a subset \( A \) of a topological space \( X \) is both \( \omega \)-open and \( \delta \omega \)-closed then it is \( \omega \)-closed.

Proof: Suppose a subset \( A \) of a topological space \( X \) is both \( \omega \)-open and \( \delta \omega \)-closed. Now \( A \subseteq A \) then by definition of \( \delta \omega \)-closed we have \( \sigma(A) \subseteq A \). Thus we have \( \sigma(A) = \sigma(A) \). Finally \( A \) is open.

Theorem 4.7

If \( X \) is an \( \delta \omega \)-closed set in \( X \), every subset of \( X \) is an \( \delta \omega \)-closed set.

Proof: Let \( X \) be topological space and open. Suppose \( A \) be any arbitrary subset of \( X \), if \( A = \emptyset \) then \( X \) is an \( \delta \omega \)-closed set in \( X \). Hence by the definition \( A \) is \( \delta \omega \)-closed. The converse is not true, which is verified by following example.

Example 4.2: Let \( X = \{b, c, d\} \) be with the topology \( \tau = \{\emptyset, \{c\}, \{d\}, \{c, d\}, X\} \). 

Thus for every subset of \( X \) is an \( \delta \omega \)-closed set, we need not be open set only if \( \{X, \phi\} \).

**Bibliography**


