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Some Enestrom- Kakeya Type Results

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Abstract: In this paper we prove some Enestrom-Kakeya type results on the location of zeros of a polynomial.

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I. INTRODUCTION

Enestrom and Kakeya proved independently a very important result on the location of zeros of a polynomial with real positive coefficients. It is known as Enestrom –Kakeya theorem and is stated as follows[\2,3]:

A. Theorem A

Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then all the zeros of P(z) lie in $|z| \leq 1$.

II. MAIN RESULTS

In this paper we change the condition on the coefficients of the polynomial and prove the following results:

A. Theorem1

Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real positive coefficients satisfying

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ...... \leq a_1 \geq a_0$$

if n is odd and

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ...... \leq a_1 \geq a_0$$

if n is even.

Then for odd n, all the zeros of P(z) lie in

$$|z| \leq \frac{2(L-M)-a_n}{a_n}$$

with $L-M > a_n$, where

$$L = a_{n-1} + a_{n-3} + ... + a_0 ,$$

$$M = a_{n-2} + a_{n-4} + ... + a_1 .$$

And for even n, , all the zeros of P(z) lie in

$$|z| \leq \frac{2(L'-M')-a_n}{a_n}$$

where

$$L' = a_{n-1} + a_{n-3} + ... + a_1 ,$$

$$M' = a_{n-2} + a_{n-4} + ... + a_2 .$$

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with $L' - M' > a_n$.

1) *Remark 1:* If we consider the polynomial

$$P(z) = z^5 + 3z^4 + 3z^3 + 4z^2 + 4z + 5,$$

then Enestrom-Kakeya theorem is not applicable.The classical Cauchy's theorem gives the bound for all the zeros of P(z) as 6, whereas by Theorem 1 , the bound is easily seen to be 3<6.

If the coefficients of the polynomial P(z) are complex , we prove the following result:

B. Theorem2

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $\operatorname{Re}(a_j) = \alpha_j$, $\operatorname{Im}(a_j) = \beta_j$, $j = 0,1,2,......,n$ and $\alpha_j > 0, \forall j$ such that

$$\alpha_n \le \alpha_{n-1} \ge \alpha_{n-2} \le \alpha_{n-3} \ge ......\ge \alpha_1 \le \alpha_0$$

if n is odd and

$$\alpha_n \le \alpha_{n-1} \ge \alpha_{n-2} \le \alpha_{n-3} \ge ......\le \alpha_1 \ge \alpha_0$$

if n is even.

Then for odd n, all the zeros of P(z) lie in

$$|z| \le \frac{2(L-M) - \alpha_n + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}$$

where

$$L = \alpha_{n-1} + \alpha_{n-3} + ...+ \alpha_0,$$

$$M = \alpha_{n-2} + \alpha_{n-4} + ...+ \alpha_1$$

with $2(L-M) - \alpha_n + 2\sum_{j=0}^{n} |\beta_j| > |a_n|$.

And for even n, , all the zeros of P(z) lie in

$$|z| \le \frac{2(L'-M') - \alpha_n + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}$$

where

$$L' = \alpha_{n-1} + \alpha_{n-3} + ...+ \alpha_1,$$

$$M' = \alpha_{n-2} + \alpha_{n-4} + ...+ \alpha_2.$$

with $2(L'-M') - \alpha_n + 2\sum_{j=0}^{n} |\beta_j| > |a_n|$.

1) *Remark 2:* Theorem 2 is a generalization of Theorem 1and reduces to it by taking $\beta_j = 0, \forall j$.

Applying Theorem 2 to the polynomial –iP(z), we get the following result:

a) *Corollary 1:* Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $\operatorname{Re}(a_j) = \alpha_j$,

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$\mathrm{Im}(a_j) = \beta_j$, $j = 0,1,2,......,n$ and $\beta_j > 0, \forall j$ such that

$$\beta_n \le \beta_{n-1} \ge \beta_{n-2} \le \beta_{n-3} \ge ......\le \beta_1 \ge \beta_0$$

if n is odd and

$$\beta_n \le \beta_{n-1} \ge \beta_{n-2} \le \beta_{n-3} \ge ......\le \beta_1 \ge \beta_0$$

if n is even.

Then for odd n, all the zeros of P(z) lie in

$$|z| \le \frac{2(L_1 - M_1) - \beta_n + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|}$$

where

$$L = \beta_{n-1} + \beta_{n-3} + ...+ \beta_0,$$
$$M = \beta_{n-2} + \beta_{n-4} + ...+ \beta_1$$

with $2(L_1 - M_1) - \beta_n + 2\sum_{j=0}^{n}|\alpha_j| > |a_n|$.

And for even n, , all the zeros of P(z) lie in

$$|z| \le \frac{2(L_1' - M_1') - \beta_n + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|}$$

where

$$L_1' = \beta_{n-1} + \beta_{n-3} + ...+ \beta_1,$$
$$M_1' = \beta_{n-2} + \beta_{n-4} + ...+ \beta_2.$$

with $2(L_1' - M_1') - \beta_n + 2\sum_{j=0}^{n}|\alpha_j| > |a_n|$.

Next we prove the following result:

C. Theorem3

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n such that for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \ j = 0,1,2,......,n$$

and $$|a_n| \le |a_{n-1}| \ge |a_{n-2}| \le |a_{n-3}| \ge ......\le |a_1| \ge |a_0|$$

if n is odd and

$$|a_n| \le |a_{n-1}| \ge |a_{n-2}| \le |a_{n-3}| \ge ......\le |a_1| \ge |a_0|$$

if n is even.

Then for odd n, all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}[|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |a_{n-3}| + ...+ |a_2|)$$

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$$-2(\cos\alpha-\sin\alpha)(|a_{n-2}|+|a_{n-4}|+......+|a_1|)+(\cos\alpha+\sin\alpha+1)|a_0|$$

and for even n, all the zeros of P(z) lie in

$$|z|\le\frac{1}{|a_n|}[|a_n|(\sin\alpha-\cos\alpha)+2(\cos\alpha+\sin\alpha)(|a_{n-1}|+|a_{n-3}|+......+|a_1|)$$

$$-2(\cos\alpha-\sin\alpha)(|a_{n-2}|+|a_{n-4}|+......+|a_2|)+(\sin\alpha-\cos\alpha+1)|a_0|$$

1) *Remark 3:* If we take $\alpha=0,\beta=0,a_j>0,\forall j$,Theorem 3 reduces to Theorem 1.

III. LEMMA

For the proofs of the above results, we need the following lemma:

A. Lemma 1

For any two complex numbers b_1,b_2 such that $|b_1|\ge|b_2|$ and $|\arg b_j-\beta|\le\alpha\le\frac{\pi}{2}, j=1,2$

for some real α,β ,

$$|b_1-b_2|\le(|b_1|-|b_2|)\cos\alpha+(|b_1|+|b_2|)\sin\alpha .$$

The above lemma is due to Govil and Rahman [1].

IV. PROOFS OF THEOREMS

A. Proof of Theorem 2

Consider the polynomial

$$F(z)=(1-z)P(z)$$

$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$

$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{\lambda+1}-a_\lambda)z^{\lambda+1}+(a_\lambda-a_{\lambda-1})z^\lambda$$

$$+......+(a_1-a_0)z+a_0$$

$$=-a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......+(\alpha_1-\alpha_0)z+\alpha_0$$

$$+i\{(\beta_n-\beta_{n-1})z^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}+......+(\beta_1-\beta_0)z+\beta_0\}.$$

For $|z|>1$ so that $\frac{1}{|z|^j}<1, j=1,2,......,n$,and odd n, we have , by using the hypothesis

$$|F(z)|\ge|a_n||z|^{n+1}-[|\alpha_n-\alpha_{n-1}||z|^n+|\alpha_{n-1}-\alpha_{n-2}||z|^{n-1}+......+|\alpha_1-\alpha_0||z|+|\alpha_0|$$

$$+|\beta_n-\beta_{n-1}||z|^n+|\beta_{n-1}-\beta_{n-2}|)|z|^{n-1}+......+|\beta_1-\beta_0||z|+|\beta_0|]$$

$$=|z|^n[|a_n||z|-\{|\alpha_n-\alpha_{n-1}|+\frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|}+\frac{|\alpha_{n-2}-\alpha_{n-3}|}{|z|^2}+.......+\frac{|\alpha_1-\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}$$

$$+|\beta_n-\beta_{n-1}|+\frac{|\beta_{n-1}-\beta_{n-2}|}{|z|}+......+\frac{|\beta_1-\beta_0|}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\}]$$

$$>|z|^n[|a_n||z|-\{|\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+|\alpha_{n-2}-\alpha_{n-3}|+.......+|\alpha_1-\alpha_0|+|\alpha_0|$$

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$$+|\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|+......+|\beta_1-\beta_0|+|\beta_0|\}]$$

$$\geq |z|^n[|a_n||z|-\{|\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+|\alpha_{n-2}-\alpha_{n-3}|+.......+|\alpha_1-\alpha_0|+|\alpha_0|$$

$$+|\beta_n|+|\beta_{n-1}|+|\beta_{n-1}|+|\beta_{n-2}|+......+|\beta_1|+|\beta_0|+|\beta_0|\}]$$

$$\geq |z|^n[|a_n||z|-\{\alpha_{n-1}-\alpha_n+\alpha_{n-1}-\alpha_{n-2}+\alpha_{n-3}-\alpha_{n-2}+......+\alpha_2-\alpha_1+\alpha_0-\alpha_1$$

$$+\alpha_0+2\sum_{j=0}^{n}|\beta_j|\}]$$

$$=|z|^n[|a_n||z|-\{2(\alpha_{n-1}+\alpha_{n-3}+......+\alpha_0)-2(\alpha_{n-2}+\alpha_{n-4}+.....\alpha_1)-\alpha_n+2\sum_{j=0}^{n}|\beta_j|\}]$$

$$=|z|^n[|a_n||z|-\{2L-2M-\alpha_n+2\sum_{j=0}^{n}|\beta_j|\}]$$

$$>0$$

if

$$|z|>\frac{2L-2M-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}$$

provided $2L-2M-\alpha_n+2\sum_{j=0}^{n}|\beta_j|>|a_n|$.

This shows that those zeros of F(z)whose modulus is greater than 1 lie in

$$|z|\leq\frac{2(L-M)-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

Since $2L-2M-\alpha_n+2\sum_{j=0}^{n}|\beta_j|>|a_n|$, it follows that those zeros of F(z) whose modulus is less than or equal to 1 already lie in

$$|z|\leq\frac{2(L-M)-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$|z|\leq\frac{2(L-M)-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}$$

in case n is odd.

If n is even, then for $|z|>1$ so that $\frac{1}{|z|^j}<1, j=1,2,......,n$, we have, as in the above case , by using the hypothesis

$$|F(z)|\geq|z|^n[|a_n||z|-\{|\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+|\alpha_{n-2}-\alpha_{n-3}|+.......+|\alpha_1-\alpha_0|+|\alpha_0|$$

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$$+|\beta_n|+|\beta_{n-1}|+|\beta_{n-1}|+|\beta_{n-2}|+......+|\beta_1|+|\beta_0|+|\beta_0|\}]$$

$$\geq |z|^n[|a_n||z|-\{\alpha_{n-1}-\alpha_n+\alpha_{n-1}-\alpha_{n-2}+\alpha_{n-3}-\alpha_{n-2}.......+\alpha_1-\alpha_2+\alpha_1-\alpha_0+\alpha_0$$

$$+2\sum_{j=0}^{n}|\beta_j|\}]$$

$$=|z|^n[|a_n||z|-\{2(\alpha_{n-1}+\alpha_{n-3}+......+\alpha_1)-2(\alpha_{n-2}+\alpha_{n-4}+......+\alpha_2)-\alpha_n+2\sum_{j=0}^{n}|\beta_j|\}]$$

$$=|z|^n[|a_n||z|-\{2(L'-M')-\alpha_n+2\sum_{j=0}^{n}|\beta_j|\}]$$

$$>0$$

if

$$|z|>\frac{2(L'-M')-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}$$

provided $2(L'-M')-\alpha_n+2\sum_{j=0}^{n}|\beta_j|>|a_n|$.

This shows that those zeros of F(z)whose modulus is greater than 1 lie in

$$|z|\leq\frac{2(L'-M')-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

Since $2(L'-M')-\alpha_n+2\sum_{j=0}^{n}|\beta_j|>|a_n|$, it follows that those zeros of F(z) whose modulus is less than or equal to 1 already lie in

$$|z|\leq\frac{2(L'-M')-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$|z|\leq\frac{2(L'-M')-\alpha_n+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}$$

in case n is even.
That completes the proof of Theorem 2.

B. Proof of Theorem 3

Consider the polynomial

$$F(z)=(1-z)P(z)$$

$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_1-a_0)z+a_0$$

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For $|z|>1$ so that $\frac{1}{|z|^{j}}<1,\ j=1,2,......,n$,and odd n, we have , as seen earlier, by using the hypothesis

$$\begin{aligned}
|F(z)| &\geq |z|^{n}[|a_n||z| - \{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + ... + |a_1 - a_0| + |a_0| \\
&\geq |z|^{n}[|a_n||z| - \{(|a_{n-1}| - |a_n|)\cos\alpha + (|a_{n-1}| + |a_n|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha \\
&\qquad + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + (|a_{n-3}| - |a_{n-2}|)\cos\alpha + (|a_{n-3}| + |a_{n-2}|)\sin\alpha \\
&\qquad + ... + (|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha + (|a_0| - |a_1|)\cos\alpha \\
&\qquad + (|a_0| + |a_1|)\sin\alpha + |a_0|\}] \\
&= |z|^{n}[|a_n||z| - \{|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |a_{n-3}| + ... + |a_2|) \\
&\qquad - 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_1|) + (\cos\alpha + \sin\alpha + 1)|a_0|\}] \\
&> 0
\end{aligned}$$

if

$$\begin{aligned}
|z| > \frac{1}{|a_n|}&[|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |a_{n-3}| + ... + |a_2|) \\
&- 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_1|) + (\cos\alpha + \sin\alpha + 1)|a_0|\}].
\end{aligned}$$

This shows that those zeros of F(z)whose modulus is greater than 1 lie in

$$\begin{aligned}
|z| \leq \frac{1}{|a_n|}&[|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |a_{n-3}| + ... + |a_2|) \\
&- 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_1|) + (\cos\alpha + \sin\alpha + 1)|a_0|\}].
\end{aligned}$$

Since those zeros of F(z)whose modulus is less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence P(z) lie in

$$\begin{aligned}
|z| \leq \frac{1}{|a_n|}&[|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |a_{n-3}| + ... + |a_2|) \\
&- 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_1|) + (\cos\alpha + \sin\alpha + 1)|a_0|\}].
\end{aligned}$$

in case n is odd.

If n is even , we have as in the above case, for $|z|>1$,

$$\begin{aligned}
|F(z)| &\geq |z|^{n}[|a_n||z| - \{(|a_{n-1}| - |a_n|)\cos\alpha + (|a_{n-1}| + |a_n|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha \\
&\qquad + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + (|a_{n-3}| - |a_{n-2}|)\cos\alpha + (|a_{n-3}| + |a_{n-2}|)\sin\alpha \\
&\qquad + ... + (|a_1| - |a_2|)\cos\alpha + (|a_1| + |a_2|)\sin\alpha + (|a_1| - |a_0|)\cos\alpha \\
&\qquad + (|a_1| + |a_0|)\sin\alpha + |a_0|\}] \\
&= |z|^{n}[|a_n||z| - \{|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |a_{n-3}| + ... + |a_1|) \\
&\qquad - 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_2|) + (\sin\alpha - \cos\alpha + 1)|a_0|\}] \\
&> 0
\end{aligned}$$

if

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$$|z| > \frac{1}{|a_n|}[|\alpha_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |v_{n-3}| + ... + |a_1|)$$
$$- 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_2|) + (\sin\alpha - \cos\alpha + 1)|a_0|\}].$$

This shows that those zeros of F(z)whose modulus is greater than 1 lie in

$$|z| \le \frac{1}{|a_n|}[|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |v_{n-3}| + ... + |a_1|)$$
$$- 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_2|) + (\sin\alpha - \cos\alpha + 1)|a_0|\}].$$

Since those zeros of F(z)whose modulus is less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence P(z) lie in

$$|z| \le \frac{1}{|a_n|}[|a_n|(\sin\alpha - \cos\alpha) + 2(\cos\alpha + \sin\alpha)(|a_{n-1}| + |v_{n-3}| + ... + |a_1|)$$
$$- 2(\cos\alpha - \sin\alpha)(|a_{n-2}| + |a_{n-4}| + ... + |a_2|) + (\sin\alpha - \cos\alpha + 1)|a_0|\}].$$

in case n is even.
That completes the proof of Theorem 3.

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