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Wardowski Type Fixed Point Theorems in Complete Metric Spaces

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***Abstract:** In this paper, we state and prove Wardowski type fixed point theorems in metric space by using a modified generalized F-contraction maps. These theorems extend other well-known fundamental metrical fixed point theorems in the literature (Dung and Hang in Vietnam J. Math. 43:743-753, 2015 and Piri and Kumam in Fixed Point Theory Appl. 2014:210, 2014, etc.).*
***Keywords:** fixed point, metric space, F-contraction.*

I. INTRODUCTION AND PRELIMINARIES

One of the most well-known results in generalizations of the Banach contraction principle is the Wardowski fixed point theorem [3]. Before providing the Wardowski fixed point theorem, we recall that a self-map T on a metric space (X,d) is said to be an F-contraction if there exist F∈Ƒ and τ∈(0,∞) such that

∀x, y∈X, [d(Tx, Ty)>0⇒τ+F(d(Tx, Ty)) ≤ F(d(x, y))],

where Ƒ is the family of all functions F:(0,∞)→R such that

(F1) F is strictly increasing, i.e. for all x, y∈R^+ such that x<y, F(x)<F(y);

(F2) for each sequence $\{x_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty$;

(F3) there exists k∈(0, 1) such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

Obviously every F-contraction is necessarily continuous. The Wardowski fixed point theorem is given by the following theorem.

A. *Theorem 1.1[3]*

Let (X, d) be a complete metric space and let T:X→X be an F-contraction. Then T has a unique fixed point x*∈X and for every x∈X the sequence $\{T(x_n)\}_{n\in N}$ converges to x^*.

Later, *Wardowski* and Van Dung [4] have introduced the notion of an *F*-weak contraction and prove a fixed point theorem for *F*-weak contractions, which generalizes some results known from the literature. They introduced the concept of an *F*-weak contraction as follows.

B. *Definition 1.2*

Let ((X, d) be a metric space. A mapping T:X→X is said to be an *F*-weak contraction on (X, d) if there exist F∈Ƒ and τ>0 such that, for all x,y∈X,

d(Tx, Ty)>0⇒τ+F(d(Tx, Ty))≤F(M(x, y)),

where

1) M(x, y)=max{d(x, y),d(x, Tx),d(y, Ty), $\frac{d(x,Ty)+d(y,Tx)}{2}$}.

By using the notion of *F*-weak contraction, Wardowski and Van Dung [4] have proved a fixed point theorem which generalizes the result of Wardowski as follows.

C. *Theorem 1.3[4]*

Let (X, d) be a complete metric space and let T:X→X be an F-weak contraction. If T or F is continuous, then T has a unique fixed point x∗∈X and for every x∈X the sequence $\{T(x_n)\}_{n\in N}$ converges to x*.

Recently, by adding values $d(T^2x, x)$, $d(T^2x, Tx)$, $d(T^2x, y)$, $d(T^2x, Ty)$ to (2), Dung and Hang [1] introduced the notion of a modified generalized *F*-contraction and proved a fixed point theorem for such maps. They generalized an *F*-weak contraction to a generalized *F*-contraction as follows.

D. *Definition 1.4*

Let (X, d) be a metric space. A mapping $T:X\to X$ is said to be a generalized *F*-contraction on (X, d) if there exist $F\in\mathcal{F}$ and $\tau>0$ such that

$\forall x, y\in X, [d(Tx, Ty)>0\Rightarrow\tau+F(d(Tx, Ty))\leq F(N(x, y))]$,

Where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x, x)+d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\}.$$

By using the notion of a generalized *F*-contraction, Dung and Hang have proved the following fixed point theorem, which generalizes the result of Wardowski and Van Dung [4].

E. *Theorem 1.5*[1]

Let (X, d) be a complete metric space and let $T:X\to X$ be a generalized F-contraction. If T or F is continuous, then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T(x_n)\}_{n\in N}$ converges to x^*.

Very recently, Piri and Kumam [2] described a large class of functions by replacing the condition (F_3) in the definition of *F*-contraction introduced by Wardowski with the following one:

(F_3'):

F is continuous on $(0, \infty)$.

They denote by F the family of all functions $F:R^+\to R$ which satisfy conditions (F_1), (F_2), and (F_3'). Under this new set-up, Piri and Kumam proved some Wardowski and Suzuki type fixed point results in metric spaces as follows.

F. *Theorem 1.6*[2]

Let T be a self-mapping of a complete metric space X into itself. Suppose there exist $F\in\mathcal{F}$ and $\tau>0$ such that

$\forall x, y\in X, [d(Tx, Ty)>0\Rightarrow\tau+F(d(Tx, Ty))\leq F(d(x, y))]$.

Then T has a unique fixed point $x^*\in X$ and for every $x_0\in X$ the sequence $\{T^n(x_0)\}_{n=1}^{\infty}$ converges to x^*.

G. *Theorem 1.7[2]*

Let T be a self-mapping of a complete metric space X into itself. Suppose there exist $F\in\mathcal{F}$ and $\tau>0$ such that

$\forall x, y\in X, [\frac{1}{2} d(x, Tx)<d(x, y)\Rightarrow\tau+F(d(Tx, Ty))\leq F(d(x, y))]$.

Then T has a unique fixed point $x^*\in X$ and for every $x_0\in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.

The aim of this paper is to introduce the modified generalized *F*-contractions, by combining the ideas of Dung and Hang [1], Piri and Kumam [2], Wardowski [3] and Wardowski and Van Dung [4] and give some fixed point result for these type mappings on complete metric space.

II. MAIN RESULTS

Let $\pounds_G$ denote the family of all functions $F:R_+\to R$ which satisfy conditions (F_1) and (F_3') and $\pounds_G$ denote the family of all functions $F:R_+\to R$ which satisfy conditions (F_1) and (F_3).

A. *Definition 2.1*

Let (X, d) be a metric space and $T:X\to X$ be a mapping. *T* is said to be modified generalized *F*-contraction of type (A) if there exist $\pounds_G$ and $\tau>0$ such that

1) $\forall x, y\in X, [d(Tx, Ty)>0\Rightarrow\tau+F(d(Tx, Ty))\leq F(M_T(x, y))]$,

where

$M_T(x, y)=\max\{d(x, y), \frac{d(x, Ty)+d(y, Tx)}{2}, \frac{d(T^2x, x)+d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)+d(x, Tx), d(Tx, y)+d(y, Ty)\}$.

B. Remark 2.2

Note that $\pounds_w$. Since, for $\beta\in(0, \infty)$, the function $F(\alpha)=\frac{-1}{\alpha+\beta}$ satisfies the conditions (F_1) and (F_3') but it does not satisfy (F_2), we have $\pounds_w$.

C. Definition 2.3

Let (X, d) be a metric space and $T:X\rightarrow X$ be a mapping. *T* is said to be modified generalized *F*-contraction of type (B) if there exist $\pounds_G$and $\tau>0$ such that
$\forall x, y\in X, [d(Tx, Ty)>0\Rightarrow\tau+F(d(Tx, Ty))\leq F(M_T(x, y))]$.

D. Remark 2.4

Note that $\pounds_w$. Since, for $\beta\in(0, \infty)$, the function $F(\alpha)=\ln(\alpha+\beta)$ satisfies the conditions (F_1) and (F_3) but it does not satisfy (F_2), we have$\pounds \subset \pounds_w$.

E. Remark 2.5

1) Every *F*-contraction is a modified generalized *F*-contraction.
2) Let *T* be a modified generalized *F*-contraction. From (3) for all $x, y\in X$ with $Tx\neq Ty$, we have

$F(d(Tx, Ty))<\tau+F(d(Tx, Ty))$

$\leq F(\max\{d(x, y), \frac{d(x, Ty)+d(y, Tx)}{2}, \frac{d(T^2x, x)+d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)+d(x, Tx), d(Tx, y)+d(y, Ty)\})$.

Then, by (F_1), we get

$d(Tx, Ty)<\max\{d(x, y), \frac{d(x, Ty)+d(y, Tx)}{2}, \frac{d(T^2x, x)+d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)+d(x, Tx), d(Tx, y)+d(y, Ty)\}$, for all $x, y\in X$, $Tx\neq Ty$.

F. Theorem 2.6

Let (X, d) be a complete metric space and $T:X\rightarrow X$ be a modified generalized F-contraction of type (A). ThenT has a unique fixed point $x^*\in X$ and for every $x_0\in X$ the sequence $\{T^n(x_0)\}_{n=1}^{\infty}$ converges to x^*.

G. Proof

Let $x_0\in X$. Put $x_{n+1}=T^n x_0$ for all $n\in N$. If, there exists $n\in N$ such that $x_{n+1}=x_n$, then $Tx_n=x_n$. That is, x_n is a fixed point of *T*. Now, we suppose that $x_{n+1}\neq x_n$ for all $n\in N$. Then $d(x_{n+1}, x_n)>0$ for all $n\in N$. It follows from (3) that, for all $n\in N$,

1) $\tau+F(d(Tx_{n-1}, Tx_n))$

$\leq F(\max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_n)+d(x_n, Tx_{n-1})}{2}, \frac{d(T^2x_{n-1}, x_{n-1})+d(T^2x_{n-1}, Tx_n)}{2}, d(T^2x_{n-1}, Tx_{n-1}), d(T^2x_{n-1}, x_n), d(T^2x_{n-1}, Tx_n)+d(x_{n-1}, Tx_{n-1}), d(Tx_{n-1}, x_n)+d(x_n, Tx_n)\})$

$=F(\max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_{n+1})+d(x_n, x_n)}{2}, \frac{d(x_{n+1}, x_{n-1})+d(x_{n+1}, x_{n+1})}{2}, d(x_{n+1}, x_{n+1}), d(x_{n+1}, x_n), d(x_{n+1}, x_{n+1})+d(x_{n-1}, x_n), d(x_n, x_n)+d(x_n, x_{n+1})\})$

$=F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})$.

If there exists $n\in N$ such that $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}=d(x_n, x_{n+1})$ then (4) becomes

$\tau+F(d(x_n, x_{n+1}))\leq F(d(x_n, x_{n+1}))$.

Since $\tau>0$, we get a contradiction. Therefore

$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}=d(x_{n-1}, x_n), \forall n\in N$.

Thus, from (4), we have

2) $F(d(x_n, x_{n+1})) \leq F(d(Tx_{n-1}, Tx_n))\leq F(d(x_{n-1}, x_n))-\tau F(d(x_{n-1}, x_n))$.

It follows from (5) and (F_1) that

$d(x_n, x_{n+1})<d(x_{n-1}, x_n),\forall n\in N$.

Therefore $\{d(x_{n+1}, x_n)\}_{n\in N}$ is a nonnegative decreasing sequence of real numbers, and hence

$\lim_{n\to\infty}d(x_{n+1}, x_n)=\gamma\geq 0$.

Now, we claim that $\gamma=0$. Arguing by contradiction, we assume that $\gamma>0$. Since $\{d(x_{n+1}, x_n)\}_{n\in N}$ is a nonnegative decreasing sequence, for every $n\in N$, we have

3) $d(x_{n+1}, x_n)\geq\gamma$.

From (6) and (F_1), we get

4) $F(\gamma)\leq F(d(x_{n+1}, x_n))\leq F(d(x_{n-1}, x_n))-\tau$

$\leq F(d(x_{n-2}, x_{n-1}))-2\tau$

$F(d(x_0, x_1))-n\tau$,

for all $n\in N$. Since $F(\gamma)\in R$ and $\lim_{n\to\infty}[F(d(x_0, x_1))-n\tau]=-\infty$, there exists $n_1\in N$ such that

5) $F(d(x_0, x_1))-n\tau<F(\gamma), \forall n>n_1$.

It follows from (7) and (8) that

$F(\gamma)\leq F(d(x_0, x_1))-n\tau<F(\gamma), \forall n>n_1$.

It is a contradiction. Therefore, we have

6) $\lim_{n\to\infty}d(x_n, Tx_n)=\lim_{n\to\infty}d(x_n, x_{n+1})=0$.

As in the proof of Theorem 2.8 in [2], we can prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. So by completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in *X*. Therefore,

7) $\lim_{n\to\infty}d(x_n,x^*)=0$.

Finally, we will show that $x^*=Tx^*$. We only have the following two cases:

a) $\forall n\in N, \exists i_n\in N, i_n>i_{n-1}, i_0=1$ and $x_{i_n+1}=Tx^*$,

$\exists\, n_3\in N, \forall n\geq n_3, d(Tx_n, Tx^*)>0$.

In the first case, we have

$x^*=\lim_{n\to\infty} x_{i_n+1}=\lim_{n\to\infty}Tx^*=Tx^*$.

In the second case from the assumption of Theorem 2.8, for all $n\geq n_3$, we have

8) $\tau+F(d(x_{n+1}, Tx^*))=\tau+F(d(Tx_n, Tx^*))$

$\leq F(\max\{d(x_n, x^*), \frac{d(x_n,Tx^*)+d(x^*,Tx_n)}{2}, \frac{d(T^2x_n,x_n)+d(T^2x_n,Tx^*)}{2}, d(T^2x_n, Tx_n), d(T^2x_n, x^*), d(T^2x_n, Tx^*)+d(x_n, Tx_n), d(Tx_n, x^*)+d(x^*, Tx^*)\})$.

From (F_3'), (10), and taking the limit as $n\to\infty$ in (11), we obtain

$\tau+F(d(x^*, Tx^*))\leq F(d(x^*, Tx^*))$.

This is a contradiction. Hence, $x^*=Tx^*$. Now, let us to show that *T* has at most one fixed point. Indeed, if $x^*,y^*\in X$are two distinct fixed points of *T*, that is, $Tx^*=x^*\neq y^*=Ty^*$, then

$d(Tx^*, Ty^*)=d(x^*, y^*)>0$.

It follows from (3) that

$F(d(x^*, y^*))<\tau+F(d(x^*, y^*))$

$=\tau+F(d(Tx^*, Ty^*))$

$\leq F(\max\{d(x^*, y^*), \frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}, \frac{d(T^2x^*,x^*)+d(T^2x^*,Ty^*)}{2}, d(T^2x^*, Tx^*), d(T^2x^*, y^*), d(T^2x^*, Ty^*)+d(x^*, Tx^*), d(Tx^*, y^*)+d(y^*, Ty^*)\})$

$=F(\max\{d(x^*, y^*), \frac{d(x^*,y^*)+d(y^*,x^*)}{2}, \frac{d(x^*,x^*)+d(x^*,y^*)}{2}, d(x^*, x^*), d(x^*, y^*), d(x^*, y^*)+d(x^*, x^*), d(x^*, y^*)+d(y^*, y^*)\})$

$=F(d(x^*, y^*))$

which is a contradiction. Therefore, the fixed point is unique.

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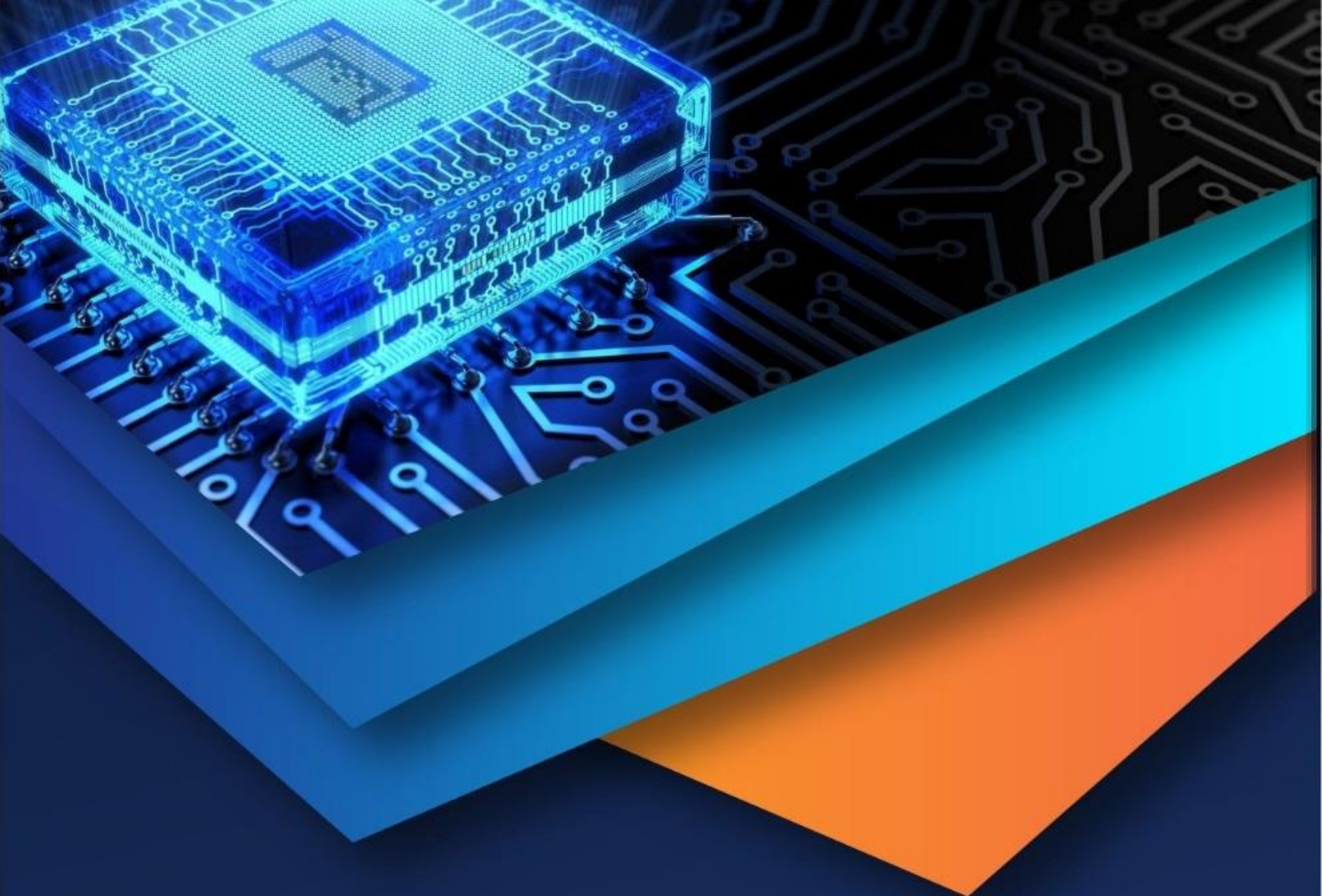

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