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A Fitted Mesh Method For Partial Differential Equations With A Time Lag

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Abstract: A class of initial boundary value problem for singularly perturbed parabolic partial differential equation of convection diffusion type having dominating delayed convection term is examined on a rectangular domain. When the perturbation parameter specifying the problem tends to zero, a breakdown of singular perturbation occurs in narrow intervals of space and short interval of time and the solution of the perturbed problem will often behave analytically quite differently. In these narrow regions which are usually referred as boundary regions, the solution changes rapidly and form parabolic boundary layers in the neighborhood of the outflow boundary regions. Due to the presence of the perturbation parameter and in particular time delay classical numerical method on uniform meshes are known to be inadequate for the solution of such type of problems, because the error in the numerical solution depends appreciably on the value of the perturbation parameter and is comparable with the solution itself for small value of perturbation parameter. Thus in connection with the stiff behavior it is of interest to develop special numerical methods whose errors would be independent of the perturbation parameter, i.e. parameter-uniformly convergent methods. In this paper, a numerical method consisting of standard upwind finite difference operator on a piecewise uniform mesh is constructed, in order to overcome the difficulties due to the presence of perturbation parameter and the time delay. The first step in this direction consists of discretizing the time variable with the backward Euler's method with constant time step. This produces a set of stationary singularly perturbed semidiscrete problem class which is further discretized in space using upwind finite difference operator on a piecewise uniform mesh. An extensive amount of analysis is carried out in order to establish the convergence and stability of the method proposed. The analysis in this paper uses a suitable decomposition of the error into smooth and singular component and a comparison principle combined with appropriate barrier functions. The error estimates are obtained which proves uniform convergence of the method.
Key Words: Convection-Diffusion Problems, Fitted Mesh, Time Delayed, Uniform Convergence, Singular Perturbation.

I. INTRODUCTION

In this paper we construct a numerical method for solving time delayed singularly perturbed convection diffusion problems involving partial differential equations, which have non smooth solutions with singularities related to the boundary layers. Such problems arise in diverse area of science and engineering that take into account not just the present state of physical systems but also its past history. These models are described by certain class of functional differential equations often called delay differential equation and from mathematical perspective they are singularly perturbed. In most application in the life sciences, a delay is introduced when there are some hidden variables and processes which are not well understood but are known to cause a time lag. The delay differential is versatile in mathematical modeling of processes in various application field, where they provide the best and sometimes the only realistic simulation of observed phenomena. The singularly perturbed differential difference equation with delay arises in general in the modeling of various real life phenomena, for instance, in studying heat or mass transfer process in composite materials with small heat conduction or diffusion, in drift diffusion model of semiconductor devices, in fluid flow problems, physiological kinetics, in various branches of biosciences and population dynamics, control theory, chemical kinetics, etc.

Some modelers ignore the "lag" effect and use an differential equations model as a substitute for a delay differential equation model. Kuang [1], comments under the heading "Small delays can have large effects" on the dangers that researchers risk if they ignore lags which they think are small, see also El'sgol's and Norkin [2]. There are inherent qualitative differences between delay differential equations and finite systems of differential equations that make such a strategy risky.

The concept of singular perturbation is not new, indeed, it has been a formidable tool in the solution of some important applied mathematical problems. A singular perturbation is a modification of partial differential equation by adding a multiple ε times a higher order term. In accordance with the informal principle, the behavior of solutions is governed primarily by the highest order terms, a solution u_ε of the perturbed problem will often behave analytically quite differently from a solution of the original equation, and exhibits boundary or the transition layers in the outflow boundary reasons when the perturbation parameter specifying the problem tends to zero [3, 4]. In these cases the primary mathematical methods fails to provide the desired solutions.

In order to obtain these solutions numerical analysis and asymptotic analysis are two principle approaches for solving these singular perturbation problems. There is a wide variety of asymptotic expansion methods available for solving the problems of above type. The method of matched asymptotic expansion [5, 6, 7], or that of multiple scales [8, 9] are available for determining as approximation, while a general procedure based on the concept of terminal value problems and its boundary layer correction has been described by O' Mally [10]. A number of authors turns their turned their attention to the question of construction asymptotic solutions one such procedure is adopted by Wasow [11] and Erdelyi [12]. But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and the outer regions, which are not routine exercises but require skill, insight and experimentation. Also asymptotic analysis gives us an asymptotic expansion of the solution, but an asymptotic expansion for the derivatives of the solutions usually cannot be derived from it. Moreover, even if we have an asymptotic expansion for the derivatives, it does not follows that this representation of the derivatives suffices for the analysis of the parameter uniform convergence in the maximum norm of its numerical solution. Then one comes down ultimately to the numerical methods.

However it is well known that serious difficulties arose [13] with numerical methods consisting of standard finite difference operators on uniform meshes, in connection with singular perturbation problems since they yield the errors which depend on an inverse power of the perturbation parameter which results in large errors for small value of the perturbation parameter [14, 15].Thus in connection with such behavior of the error the quest for parameter uniform methods found special relevance.

For certain problems having sufficiently smooth solutions, ε-uniform numerical methods which converge irrespective of ε have been developed and thoroughly studied [16, 4, 13, 17, 18, 3, 19, 20] with the convergence analysis given in the maximum norm. The literature cited here is only small part of it and lot worth mention, for instance, more recently, the parameter uniform methods for convection diffusion problems are well treated in literature by several authors, for instance, Shishkin et. al. [21], on the basis of posteriori adaptive mesh parameter uniform scheme for problems having moving interior layers is constructed, for a class of singularly perturbed partial differential equation of convection diffusion type. Dunne et. al. in [22], proposed a fitted mesh method consisting of upwind finite difference operator, for a class of time dependent convection diffusion problem with a boundary turning point, on a rectangular domain. While the singularly perturbed convection diffusion problem with two small parameters is considered by Roos et. al. in [23], parameter uniform convergence using the streamline diffusion finite element method on Shishkin mesh is shown in this paper. Wang [24] present a convergence analysis, for the exponentially fitted finite volume method in two dimension applied to a class of singularly perturbed convection diffusion equation with exponential boundary layers, which is then shown to be independent of the perturbation parameter.Another attempt is due to $Lin\beta$ [25], in this paper singularly perturbed convection diffusion problem with a concentrated source and a discontinuous convection field is considered, using two upwind difference schemes on general meshes the epsilon uniform convergence of scheme is proved in the maximum norm on Shishkin and Bakhvalov meshes. A method of defect correction technique is also used by several authors for constructing the parameter uniform methods of higher order accuracy for convection diffusion problems [26, 27, 28], Hemker [29] presents an ε-uniform convergent scheme of higher order accuracy in time and space, based on the defect correction principle, in the case of boundary value problems for singularly perturbed parabolic convection diffusion equations.

The fact that singular perturbation problems involves partial differential equation is of crucial importance as many additional technical issues arise with partial differential equations, such as the smoothness of the solution, the compatibility of the data, and the geometry of the domain. In fact, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by partial differential equation, the study of partial differential equations is of great practical value. But there is no general theory known concerning the solvability of all partial differential equations, instead, research focuses on various particular partial differential equations that are important for applications, within and outside of mathematics.

In spite of their great practical value and applications, there does not seem to have been any attempt to study time dependent convection diffusion problems in a general setting with a retarded argument. Our aim in this paper is to carry out such a study, at any rate in connection with the singular perturbation.

A. Notations And Terminology

Most of the notations and symbols we employ are fairly standard. Throughout the paper C will denote a generic positive constant which may take different values in different equations and inequalities, but are always independent of Δt, Δx and ε and in case of discrete problems, also independent of the mesh parameters. Also $\|\cdot\|$ denotes the global maximum norm over the domain of the independent variable, i.e, $\| f \| = max_{\overline{D}} |f(x,t)|$.

II. CONTINUOUS PROBLEM

Consider the following class of singularly perturbed parabolic partial differential equation of convection diffusion type having time delayed convection term

$$L_\varepsilon u(x,t) \equiv u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x)u_x(x,t-\tau) + b(x)u(x,t) = f(x,t) \qquad (1)$$

in the region

$$D \equiv \{(x,t) \in \Omega \times (0,T] \equiv (0,1) \times (0,T]\}.$$

With initial condition

$$u(x,s) = u_0(x,s) \qquad (2)$$

where $x \in \Omega \equiv (0,1)$ and $s \in [-\tau, 0)$. Subject to the boundary conditions

$$u(0,t) = u(1,t) = 0 \qquad (3)$$

where $t \in (0,T]$, and ε is the singular perturbation parameter, $0 < \varepsilon << 1$ and τ is the delayed parameter so that $0 \le \tau = o(\varepsilon^2)$ and the prescribed function $f(x,t)$, a(x) and b(x) are sufficiently smooth.

However, for $\varepsilon \to 0$, the solution of the problem typically exhibits layer behavior depending on the sign of the time delayed convection term. In this paper the delayed convection diffusion problem is considered, which exhibits only boundary layer behavior i.e. $a(x) \ge \beta > 0$, where β is some positive constant and also $b(x) \ge \theta > 0$ can be assumed without loss of generality on $\overline{D}$ for some constant θ, since this can be easily obtained by a change of variable independent of the perturbation parameter. The investigation here is to examine the solution when the delay is non-zero and the effect of the delay on the boundary layer solution.

III. ANALYTICAL RESULTS

A. *Lemma 3.1* Let $\Phi(x,t)$, $\Psi(x,t) \in C^{2,1}$ i.e., twice continuously differential in space and once in time, and satisfy the relation $|L[\Phi]| < -L[\Psi]$ where L is the operator defined in (1). If $|\Phi| \le \Psi$ holds on the boundary, then $|\Phi| \le \Psi$ holds through out the domain.

B. *Theorem 3.2* There exist a number C independent of the perturbation parameter ε such that for all sufficiently small positive values of ε, $|u(x,t) - \Phi(x)| \le Ct, \quad t \in (0,T]$.

Furthermore $|u(x,t)| \le C, \quad (x,t) \in \Omega \times (0,T]$.

C. *Lemma 3.3* By keeping x fixed along the line $\{(x,t): 0 \le t \le T\}$ the bound of u_t is given by

$$|u_t(x,t)| \le C.$$

Furthur

$$\left|\frac{\partial^i u(x,t)}{\partial t^i}\right| \le C \quad \text{for} \quad i = 0,1,2,3$$

Proof. We assume that the solution $u(x,t)$ is sufficiently smooth in the domain $\overline{\Omega} \times [0,T]$ and by mean value theorem there exits a t^* in the interval $(t, t+k)$ along the line $\{(x,t): 0 \le t \le T\}$ such that

$$u_t(x,t^*) = \frac{u(x,t+k) - u(x,t)}{k} \qquad (4)$$

$$|u_t(x,t^*)| \le \frac{2|u(x,t)|}{k}$$

using the equation (??), we get

$$|u_t(x,t)| \le C \qquad (5)$$

Similarly we get the bounds of the $u_{tt}(x,t)$ and $u_{ttt}(x,t)$ along the line $\{(x,t): 0 \le t \le T\}$, i.e. we have

$$\left|\frac{\partial^i u(x,t)}{\partial t^i}\right| \le C \quad \text{for} \quad i = 0,1,2,3 \qquad (6)$$

■

IV. THE TIME SEMIDISCRETIZATION

Writing the equation in the form,

$$u_t = \varepsilon u_{xx} - a(x)u_x(x,t-\tau) - b(x)u + f(x,t) \qquad (7)$$

We discretize the time variable by mean of the Euler's implicit rule with uniform stepsize Δt,

$$u(x,0) = u_0$$

$$\frac{u_{j+1} - u_j}{\Delta t} = [\varepsilon u_{xx} - a(x)u_x(x,t-\tau) - b(x)u + f]_{j+1}$$

$$= \varepsilon[u_{j+1}]_{xx} - a(x)[u_{j-k+1}]_x - b(x)u_{j+1} + f(x, t_{j+1}). \quad (8)$$

i.e.

$$u_j = u_{j+1} - \varepsilon\Delta t[u_{j+1}]_{xx} + a(x)\Delta t\ [u_{j-k+1}]_x + b(x)\Delta t\ u_{j+1} - \Delta t\ f(x, t_{j+1}) \quad (9)$$

with boundary condition

$$u_{j+1}(0) = 0, \quad u_{j+1}(1) = 0. \quad (10)$$

for $j = 0, \dots, M-1$.

We may write,

$$-\varepsilon\Delta t\ [u_{j+1}]_{xx} + (1 + \Delta t\ b(x))\ u_{j+1} = h(x) \quad (11)$$

i.e.

$$L_{x,\varepsilon} u_{j+1} = h(x)$$

with boundary condition

$$u_{j+1}(0) = 0, \quad u_{j+1}(1) = 0. \quad (12)$$

for $j = 0, \dots, M-1$, where

$$L_{x,\varepsilon} = -\varepsilon\Delta t\frac{\partial}{\partial x^2} + (I + \Delta t\ b(x)) \quad (13)$$

$$h(x) = u_j - a(x)\Delta t[u_{j-k+1}]_x + \Delta t f(x, t_{j+1}). \quad (14)$$

Now, since the solution of (4) is smooth enough, it holds

$$\begin{aligned} u(t_j) &= u(t_{j+1} - \Delta t) \\ &= u(t_{j+1}) - \Delta t\ \frac{\partial u}{\partial t}(t_{j+1}) + \int_{t_j}^{t_{j+1}} (t_j - \xi)\frac{\partial^2 u}{\partial t^2}(\xi)\ d\xi \\ &= u(t_{j+1}) - \Delta t(\varepsilon u_{xx} - a(x)u_x(x, t - \tau) - b(x)u + f(x, t)) + O(\Delta t^2) \end{aligned} \quad (15)$$

subtracting (4) from (??), and note that the local truncation error $e_{j+1} \equiv u(t_{j+1}) - \hat{u}_{j+1}$ at $(j + 1)^{th}$ time step is the solution of boundary value problem of type

$$L_{x,\varepsilon} e_{j+1} = O(\Delta t^2) \quad (16)$$

$$e_{j+1}(0) = e_{j+1}(1) = 0. \quad (17)$$

where $\hat{u}_{j+1}$ is the solution of the boundary value problem of semidiscretized problem class (4). Now, the operator $L_{x,\varepsilon}$ clearly satisfies the maximal principle, consequently

$$\| L_{x,\varepsilon}^{-1} \|_\infty \le C.$$

Which ensures the stability of the semidiscretized scheme, where C is the positive constant independent of Δt. Thus we have

$$\| e_{j+1} \|_\infty \le C(\Delta t)^2. \quad (18)$$

We have proved the following

A. Lemma 4.1 If $\left|\frac{\partial^i}{\partial t^i} u(x, t)\right| \le C$, for $(x, t) \in \bar{\Omega} \times (0, T]$ and $0 \le i \le 2$. Then the local truncation error satisfies the estimate

$$\| e_{j+1} \|_\infty \le C(\Delta t)^2.$$

■

Now, using the local error estimates upto the $(j + 1)^{th}$ time step, followed by classical combination of stability and consistency property of the scheme leads us to,

B. Theorem 4.2 Under the hypothesis of the preceding lemma, the global truncation error satisfies the estimate

$$\| E_j \|_\infty \le C\Delta t. \quad (19)$$

Proof.

$$\begin{aligned} \| E_j \|_\infty &= \| \sum_{k=1}^{j} e_k \|_\infty \\ &\le \| e_1 \|_\infty + \| e_2 \|_\infty + \cdots + \| e_j \|_\infty \\ &\le C\ T\Delta t \\ &= C\Delta t, \end{aligned} \quad (20)$$

since $j\Delta t \le T$. Where C is a positive constant independent of ε and Δt.

■

Which ensures, first order uniform convergence of the time semidiscritization process.

V. SOME A-PRIORI FOR SEMIDISCRETIZED PROBLEM

In this section of the paper we discuss qualitative properties of the solution of the semidiscretized problem, giving a-priori information that can be used advantageously in the development of the numerical approximations. An intuitive idea about the behavior of the semidiscretized problem is obtained by associating with this equation some physical interpretations. We now give to the maximal principle a mathematical basis, that can be a useful tool for deriving a priori bounds on the solutions of the differential equations and their derivatives. More background may be found in Protter and Weinberger [30] and Sperb [31] for a comprehensive discussion of these principles. The differential operator L_ε satisfies the following maximum principle.

*A. Lemma 5.1 (Maximum Principle)*If a function Φ, twice continuously differentiable in space and once in time, i.e. $\Phi\in C^{2,1}(\Omega)$ be such that $\Phi(x,t)\geq 0$, for all $(x,t)\in\partial D$ Then $L_\varepsilon\Phi(x,t)\geq 0$ for all $(x,t)\in D$ implies that $\Phi(x,t)\geq 0$ for all $(x,t)\in\bar{D}$.

Proof. Let (x^*,t^*) be such that $\Phi(x^*,t^*)=\min_{x\in\bar{D}}\Phi(x,t)$, and suppose that $\Phi(x^*,t^*)<0$. It is then clear from the hypothesis that $(x^*,t^*)\notin\partial D$, from this it follows $(x^*,t^*)\in D$, hence $\Phi_x(x^*,t^*)=0$, and $\Phi_{xx}(x^*,t^*)\geq 0$,

$$\begin{aligned} L_{x,\varepsilon}\Phi(x^*,t^*) &= -\varepsilon\Delta t\Phi_{xx}(x^*,t^*)+(I+\Delta t b(x))\Phi(x^*,t^*)\\ &\leq 0 \end{aligned} \tag{21}$$

which is a contradiction to our assumption, hence our assumption $\Phi(x^*,t^*)<0$ is wrong. Hence, $\Phi(x,t)\geq 0$ for all $x\in\bar{D}$. Hence the result holds.

■

In the next two lemmas we establish a priori bound on the exact solution and its derivatives, of time semidiscretized problem class (4),

B. Lemma 5.2 Let u_ε be the solution of any problem from semidiscretized problem class (4), then

$$\| u_\varepsilon \|_\infty \leq C. \tag{22}$$

Proof. Consider the functions

$$\zeta^\pm = \frac{1}{\theta}\ \| h \|_\infty \pm u_\varepsilon(x,t). \tag{23}$$

Note that the functions $\zeta^\pm$ are nonnegative at x=0,1 for all $t\in(0,T]$, and that for all (x,t)

$$\begin{aligned} L_{x,\varepsilon}\zeta^\pm(x,t) &= \left\{-\varepsilon\Delta t\frac{\partial^2}{\partial x^2}+(I+\Delta t b(x))\right\}\left(\frac{1}{\theta}\ \| h \|_\infty \pm u_\varepsilon(x,t)\right)\\ &= \frac{1}{\theta}\ \| h \|_\infty + \frac{b(x)\Delta t}{\theta}\ \| h \|_\infty \pm L_{x,\varepsilon}u_\varepsilon(x,t)\\ &= \frac{1}{\theta}\ \| h \|_\infty + \frac{b(x)\Delta t}{\theta}\ \| h \|_\infty \pm h(x)\\ &\geq 0, \end{aligned} \tag{24}$$

since $b(x)\geq\theta>0\Rightarrow b(x)\theta^{-1}\geq 1$ and $\| h \|_\infty\geq h(x)$. Thus maximal principle gives $\zeta^\pm(x,t)\geq 0$ and so,

$$\| u_\varepsilon \|_\infty \leq \frac{1}{\theta}\ \| h \|_\infty \quad \text{for all } (x,t)\in\bar{D}.$$

■

C. Remark 5.3 Uniqueness of the solution follows from lemma (5.1), existence follows from uniqueness as the problem under consideration is linear, and the boundedness of the solution is implied by the lemma (5.2).

■

D. Lemma 5.4 Let u_ε be the solution of any problem from semidiscretized problem class (4), then

$$|u_\varepsilon^k| \leq C(1+\varepsilon^{-\frac{k}{2}}E(x,\theta)). \tag{25}$$

Bounds for the derivative of u_ε were first obtained obtained by O'Riordan et. al. [32] , using the technique based on those of kellogg et. al. [33]. However, stronger results than these are required to obtain the ε-uniform convergence results in this paper.

DECOMPOSITION

The solution u_ε of the semidiscrete problem is decomposed into a smooth component v_ε and a singular component w_ε as follows :

$$u_\varepsilon = v_\varepsilon + w_\varepsilon$$

where

$$v_\varepsilon = v_0 + \varepsilon v_1,$$

where v_0 is the solution of the reduced problem i.e. by keeping $\varepsilon=0$ in the semidiscritized problem and w_ε is the solution of the homogeneous problem

$$L_{x,\varepsilon}w_\varepsilon = 0,$$
$$w_\varepsilon(0)=u_0-v_0(0) \quad \text{and} \quad w_\varepsilon(1)=u_1-v_0(1).$$

and consequently v_1 satisfies

$$L_{x,\varepsilon} v_1 = v_0'',$$
$$v_1(0) = 0, \quad v_1(1) = 0$$

Lemma 5.5 At each time step $(j+1)$, the smooth component $(v_\varepsilon)_{j+1}$ of the decomposition of the solution u_ε of the semidiscrete problem, for $0 \le k \le 3$ satisfies

$$|(v_\varepsilon)_{j+1}^{k}(x)| \le C(1 + \varepsilon^{-(k-2)/2}\ E(x,\theta)) \tag{26}$$

for all $x \in \overline{\Omega}$, where

$$E(x,\theta) = e^{-x\sqrt{\frac{\theta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\theta}{\varepsilon}}}.$$

Proof. Because of the bound on v_0'' it is clear that v_1 is the solution of a problem similar to the semidiscretize problem (4), this implies that for $0 \le k \le 3$

$$|(v_1^k)_{j+1}(x)| \le C(1 + \varepsilon^{-(k-2)/2}\ E(x,\theta)).$$

Since,

$$u_\varepsilon^k = v_\varepsilon^k + w_\varepsilon^k$$

the above estimates for v_0^k, v_1^k, and w_ε^k yields, for $0 \le k \le 3$

$$|(v_\varepsilon)_{j+1}^{k}(x)| \le C(1 + \varepsilon^{-(k-2)/2}\ E(x,\theta))$$

for all $x \in \overline{\Omega}$.

Lemma 5.6 At each time step $(j+1)$, the singular component $(w_\varepsilon)_{j+1}$ of the decomposition of the solution u_ε of the semidiscrete problem, for $0 \le k \le 3$ satisfies

$$|((w_\varepsilon)^k)_{j+1}(x)| \le C\ \varepsilon^{-k/2}\ E(x,\theta) \tag{27}$$

for all $x \in \overline{\Omega}$, where

$$E(x,\theta) = e^{-x\sqrt{\frac{\theta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\theta}{\varepsilon}}}.$$

Proof. Introducing the barrier function,

$$\Psi_{j+1}^{\pm}(x) = CE(x,\theta) \pm (w_\varepsilon)_{j+1}(x) \tag{28}$$

where C is arbitrary constant chosen sufficiently large so that, following inequality follows

$$\Psi_{j+1}^{\pm}(0) \ge 0, \quad \Psi_{j+1}^{\pm}(1) \ge 0 \tag{29}$$

and

$$\begin{aligned} L_{x,\varepsilon}\Psi_{j+1}^{\pm}(x) &= \left\{-\varepsilon\Delta t\frac{\partial^2}{\partial x^2} + (I + \Delta t b(x))\right\}(CE(x,\theta) \pm (w_\varepsilon)_{j+1}(x)) \\ &= -C\varepsilon\Delta t\frac{\theta}{\varepsilon}E(x,\theta) + (I + \Delta t b(x))CE(x,\theta) \pm L_{x,\varepsilon}(w_\varepsilon)_{j+1}(x)) \\ &\ge 0 \end{aligned} \tag{30}$$

since $b(x) \ge \theta > 0$ and by the fact that singular component of the decomposition satisfies the corresponding homogeneous problem. The maximum principle then gives

$$|(w_\varepsilon)_{j+1}(x)| \le CE(x,\theta) \tag{31}$$

for all $x \in \overline{\Omega}$. To bound the first derivative Since the singular component satisfies $L_{x,\varepsilon}(w_\varepsilon)_{j+1} = 0$, it follows that $L_{x,\varepsilon}((w_\varepsilon)_{j+1})' = -b'\Delta t(w_\varepsilon)_{j+1}$. Now introduce the barrier functions

$$\psi_{j+1}^{\pm}(x) = \frac{C}{\sqrt{\varepsilon}}E(x,\theta) \pm ((w_\varepsilon)_{j+1})'(x) \tag{32}$$

where C is arbitrary constant chosen sufficiently large so that, following holds,

$$\psi_{j+1}^{\pm}(0) = \frac{C}{\sqrt{\varepsilon}}E(0,\theta) \pm ((w_\varepsilon)_{j+1})'(0) \ge 0, \tag{33}$$

$$\psi_{j+1}^{\pm}(1) = \frac{C}{\sqrt{\varepsilon}}E(1,\theta) \pm ((w_\varepsilon)_{j+1})'(1) \ge 0, \tag{34}$$

and

$$\begin{aligned} L_{x,\varepsilon}\psi_{j+1}^{\pm}(x) &= \left\{-\varepsilon\Delta t\frac{\partial^2}{\partial x^2} + (I + \Delta t b(x))\right\}\left(\frac{C}{\sqrt{\varepsilon}}E(x,\theta) \pm ((w_\varepsilon)_{j+1})'(x)\right) \\ &= \left\{-\varepsilon\Delta t\frac{\partial^2}{\partial x^2} + (I + \Delta t b(x))\right\}\frac{C}{\sqrt{\varepsilon}}E(x,\theta) \pm L_{x,\varepsilon}((w_\varepsilon)_{j+1})' \end{aligned}$$

$$\geq \frac{C\Delta t}{\sqrt{\varepsilon}}E(x,\theta)(b(x)-\theta)+\frac{C}{\sqrt{\varepsilon}}E(x,\theta)\mp b'\Delta tCE(x,\theta)$$
$$=\frac{C\Delta t}{\sqrt{\varepsilon}}E(x,\theta)(b(x)-\theta)+CE(x,\theta)(\frac{1}{\sqrt{\varepsilon}}\mp b'\Delta t)$$
$$\geq 0, \tag{35}$$

for sufficiently small ε and large C.
The maximum principle then yields,

$$|((w_\varepsilon)_{j+1})'|\leq \frac{C}{\sqrt{\varepsilon}}E(x,\theta). \tag{36}$$

The estimates for the second and the third derivative of the singular component w_{j+1}^{ε} can easily be obtained from the differential equation and the estimates of w_{j+1}^{ε}, and $(w_{j+1}^{\varepsilon})'$. Thus, for $0\leq k\leq 3$

$$((w_\varepsilon)_{j+1})^k\leq C\ \varepsilon^{\frac{-k}{2}}E(x,\theta). \tag{37}$$

VI. THE SPATIAL DISCRETIZATION

When we apply a finite difference method to a singular perturbation problem, with boundary layers in its solution, the question of choice of the mesh immediately arises. Since we allow the use of a simple interpolent in the case of finite difference methods to capture the boundary layer we must place some of the mesh point in the interior of the boundary layer region. In case of the uniform meshes none of the mesh point is in the boundary layer region unless N is of order $1/\varepsilon$, that seems to be too restrictive, and that leads one to decide what kind of non uniform mesh is to use then. It is often sufficient to construct a piecewise uniform mesh, that is adapted to the singular perturbation, such meshes are referred in general as fitted meshes. The piecewise uniform fitted meshes were first introduced by Shishkin [34]. The first numerical result using a fitted mesh method were presented in Miller et. al. [35]. Different approaches to adapting the mesh, involving complicated redistribution of the mesh points, have been taken by several authors, Bakhvalov [16], Gartland [36], Liseikin [37], and Vulanovic′ [38] but none has the simplicity of the piecewise uniform fitted mesh. So we are proceeding further with piecewise uniform mesh, that is a mesh composed of a uniform fine mesh in the boundary layer and a uniform coarse mesh outside the boundary layer region having different mesh parameters.
In this paper we are considering the upwind discretization on piecewise uniform fitted meshes, i.e. the derivative term at a given mesh point is approximated by a discrete derivative, which uses the mesh points only in the upwind direction from that mesh point, also, it reduces the truncation error. Since in practice we often want to construct a numerical method whose system matrix is a M-matrix. Much has been written in the literature about the pros and cons of upwind discretization. The interested reader may consult Roache [39], or Gresho and Lee [40].

A. *Fitted Mesh Finite Difference Method*

In this section, we discretize the boundary value problem (4) using the fitted mesh finite difference method composed of a standard upwind finite difference operator on a piecewise uniform mesh condensing at the boundary points. The fitted piecewise uniform mesh $\overline{\Omega}_x^N=\{x_i\}_{i=0}^N$ on the interval $[0,1]$, is constructed by partitioning the interval into three subintervals, $(0,\delta)$, $(\delta,1-\delta)$ and $(1-\delta,1)$. on each of the subintervals, a uniform mesh is placed, i.e., the interval $(0,\delta)$ and $(1-\delta,1)$ are each divided into $N/4$ equal mesh points and the interval $(\delta,1-\delta)$ is divided into $N/2$ equal mesh points. The resulting piecewise uniform mesh depends on one parameter which is called the transition parameter δ and is chosen such that

$$\delta\equiv \min[1/4, C\ \varepsilon\Delta t\ \ln N], \tag{38}$$

where C is the constant whose value depends upon the method applied.
Define

$$x_i=\begin{cases} ih_i, & \text{for} 0\leq i\leq N/4\\ \delta+(i-N/4)h_i, & \text{for} N/4+1\leq i\leq 3N/4\\ 1-\delta+(i-3N/4)h_i, & \text{for} 3N/4+1\leq i\leq N\end{cases} \tag{39}$$

where N is the number of mesh points.

$$h_i=\begin{cases} 4\delta/N, & \text{for} 0\leq i\leq N/4\\ 2(1-2\delta)/N, & \text{for} N/4+1\leq i\leq 3N/4\\ 4\delta/N, & \text{for} 3N/4+1\leq i\leq N.\end{cases} \tag{40}$$

The fitted mesh finite difference method for the time semidiscretized problem (4) at $(j+1)^{th}$ time step on the piecewise uniform mesh $\overline{\Omega}^N$ is defined by

$$L^N_{x,\varepsilon}u_{i,j+1} = h(x_i) \quad (41)$$

$$u_{0,j+1} = u_{N,j+1} = 0 \quad (42)$$

where the discrete operator is defined as

$$L^N_{x,\varepsilon}u_{i,j+1} = -\varepsilon\Delta t D^+D^-u_{i,j+1} + (I + \Delta t b(x_i))u_{i,j+1} \quad (43)$$

with

$$D^+D^-u_{i,j+1} = \frac{2(D^+u_{i,j+1} - D^-u_{i,j+1})}{(h_i + h_{i+1})} \quad (44)$$

$$D^-u_{i,j+1} = \frac{(u_{i,j+1} - u_{i-1,j+1})}{h_i} \quad (45)$$

$$D^+u_{i,j+1} = \frac{(u_{i+1,j+1} - u_{i,j+1})}{h_{i+1}}. \quad (46)$$

1) *Remark 6.1* At first sight, the parameter Δt seems to behave like another pertubation parameter into the time semidiscretized problem, but under enough smoothness and compatibility requirements on the data of the continuous problem we have seen that this parameter does not have any severe effect as that of the perturbation parameter ε, to the multiscale character of the semidiscrete solutions.

VII. CONVERGENCE AND STABILITY ANALYSIS

In investigating the differece schemes, primary considerations is given to the fundamental question of stability of the difference solution with respect to the perturbation parameter. by now the most complete results have been obtained for computational methods approximating the linear problems of mathematical physics Samarskii [?], Samarskii and Gulin [41] and Thomee [42]. On the basis of the stability estimates one can trivially study the convergence rate of the difference scheme as well.

A. *Stability Analysis*

A discrete maximum principle can also be established directly, without appealing to the properties of the elements in the system matrix. Analogously to the continuous case, this is usually proved by the method of contradiction as we now show

1) *Theorem 7.1* (Discrete Maximum Principle) Assume that mesh function $\Psi_{i,j+1}$ satisfies $\Psi_{0,j+1} \geq 0$ and $\Psi_{N,j+1} \geq 0$. Then $L^N_{x,\varepsilon}\Psi_{i,j+1} \geq 0$ for all $x_i \in \Omega^N_x$ implies that $\Psi_{i,j+1} \geq 0$ for all $x_i \in \overline{\Omega}^N_x$.

Proof. Assume that there exists a positive integer k such that $\Psi_{k,j+1} = \min_{0\leq i\leq N}\Psi_{i,j+1}$ and assume $\Psi_{k,j+1} < 0$. Then we have $\Psi_{0,j+1} \geq 0$ and $\Psi_{N,j+1} \geq 0$, therefore it follows from the hypothesis that $k \notin \{0, N\}$ also $\Psi_{k,j+1} - \Psi_{k-1,j+1} \leq 0$ and $\Psi_{k+1,j+1} - \Psi_{k,j+1} \geq 0$ and therefore

$$\begin{aligned}
L^N_{x,\varepsilon}\Psi_{k,j+1} &= -\varepsilon\Delta t D^+D^-\Psi_{k,j+1} + (I + \Delta t b(x_i))\Psi_{k,j+1} \\
&= -\varepsilon\Delta t\left[\frac{\Psi_{k-1,j+1} - 2\Psi_{k,j+1} + \Psi_{k+1,j+1}}{h^2}\right] + (1 + \Delta t b(x_i))\Psi_{k,j+1} \\
&= \frac{-2\varepsilon\Delta t}{h_{k,j+1} + h_{k+1,j+1}}\left[\frac{\Psi_{k+1,j+1} - \Psi_{k,j+1}}{h_{k+1}} - \frac{\Psi_{k,j+1} - \Psi_{k-1,j+1}}{h_k}\right] + (I + \Delta t b(x_i))\Psi_{k,j+1} \\
&< 0,
\end{aligned} \quad (47)$$

since $\Psi_{k,j+1} < 0$ by assumption and $(I + \Delta t b(x_i)) \geq \tilde{\theta} \geq 0$. Which is a contradiction to our assumption, therefore the assumption $\Psi_{k,j+1} < 0$ is wrong, hence $\Psi_{k,j+1} \geq 0$. We have chosen k fixed and arbitrarily, so $\Psi_{i,j+1} \geq 0$ for all $x_i \in \overline{\Omega}^N_x$.

2) *Lemma 7.2* Let $Z_{i,j+1}$ be any mesh function such that $Z_{0,j+1} = Z_{N,j+1} = 0$ then for all i, $0 \leq i \leq N$

$$|Z_{i,j+1}| \leq \theta^{-1} \max_{1\leq k\leq N-1} |L^N_{x,\varepsilon}Z_{k,j+1}|. \quad (48)$$

Proof. To prove the estimate we construct two mesh function

$$\Psi^{\pm}_{i,j+1} = \theta^{-1} \max_{1\leq k\leq N-1} |L^N_{x,\varepsilon}Z_{k,j+1}| \pm Z_{i,j+1} \quad (49)$$

clearly we have

$$\Psi^{\pm}_{0,j+1} = \theta^{-1} \max_{1\leq k\leq N-1} |L^N_{x,\varepsilon}Z_{k,j+1}| \pm Z_{0,j+1} \geq 0 \quad (50)$$

$$\Psi^{\pm}_{N,j+1} = \theta^{-1} \max_{1\leq k\leq N-1} |L^N_{x,\varepsilon}Z_{k,j+1}| \pm Z_{N,j+1} \geq 0 \quad (51)$$

and for $1 \leq i \leq N - 1$

$$\begin{aligned} L^N_{x,\varepsilon}\Psi^{\pm}_{i,j+1} &= -\varepsilon\Delta t D^+D^-\Psi^{\pm}_{i,j+1} + (I + \Delta t b(x_i))\Psi^{\pm}_{i,j+1} \\ &= \theta^{-1}(I + \Delta t b(x_i)) \max_{1\le k\le N-1} |L^N_{x,\varepsilon}Z_{k,j+1}| \pm L^N_{x,\varepsilon}Z_{i,j+1} \\ &\geq \max_{1\le k\le N-1} |L^N_{x,\varepsilon}Z_{k,j+1}| \pm L^N_{x,\varepsilon}Z_{i,j+1} \\ &\geq 0, \end{aligned} \tag{52}$$

since $(I + \Delta t b(x_i)) \geq \tilde{\theta} > 0$.

Thus discrete maximum principle gives us

$$\Psi^{\pm}_{i,j+1} \geq 0, \quad 0 \le i \le N,$$

i.e.

$$\theta^{-1} \max_{1\le k\le N-1} |L^N_{x,\varepsilon}Z_{k,j+1}| \pm Z_{i,j+1} \geq 0$$

which implies,

$$|Z_{i,j+1}| \le \theta^{-1} \max_{1\le k\le N-1} |L^N_{x,\varepsilon}Z_{k,j+1}|.$$

which proves the required estimates.

B. *Convergence Analysis*

1) *Decomposition:* At each time step $(j+1)$ the solution u^N_ε of the discrete problem are decomposed in a similar manner to the decomposition of the solution u_ε of the continuous problem (1), (2). Thus

$$u^N_\varepsilon(x_{i,j+1}) = v^N_\varepsilon(x_{i,j+1}) + w^N_\varepsilon(x_{i,j+1})$$

where v^N_ε is the solution of the inhomogeneous problem

$$L^N_{x,\varepsilon}v^N_\varepsilon(x_{i,j+1}) = f(x_{i,j+1}),$$
$$v^N_\varepsilon(x_{0,j+1}) = 0 \quad \text{and} \quad v^N_\varepsilon(x_{1,j+1}) = 0.$$

and $w^N_\varepsilon(x_{i,j+1})$ is the solution of the homogeneous problem

$$L^N_{x,\varepsilon}w^N_\varepsilon(x_{i,j+1}) = 0,$$
$$w^N_\varepsilon(x_{0,j+1}) = 0 \quad \text{and} \quad w^N_\varepsilon(x_{1,j+1}) = 0.$$

Then the error at each time step (j+1) can be written in the form

$$(u^N_\varepsilon - u_\varepsilon)(x_{i,j+1}) = (v^N_\varepsilon - v_\varepsilon)(x_{i,j+1}) + (w^N_\varepsilon - w_\varepsilon)(x_{i,j+1}).$$

Now in the next two result we estimates the error in the smooth and singular component respectively.

2) *Error in Smooth Component*

Lemma 7.3 At each time step $(j+1)$ and mesh point $x_i \in \overline{\Omega}^N_x$ the smooth component of error satisfies

$$|(v^N_\varepsilon - v_\varepsilon)(x_{i,j+1})| \le C\ N^{-1}M^{-1} \tag{53}$$

Proof. This is obtained using the following standard stability and consistency argument, we consider the local truncation error,

$$\begin{aligned} L^N_{x,\varepsilon}(v^N_\varepsilon - v_\varepsilon)(x_{i,j+1}) &= (L^N_{x,\varepsilon}v^N_\varepsilon - L^N_{x,\varepsilon}v_\varepsilon)(x_{i,j+1}) \\ &= (f - L^N_{x,\varepsilon}v_\varepsilon)(x_{i,j+1}) \\ &= (L_{x,\varepsilon} - L^N_{x,\varepsilon})v_\varepsilon(x_{i,j+1}) \\ &= -\varepsilon\Delta t\left(\frac{\partial^2}{\partial x^2} - D^+D^-\right)v_\varepsilon(x_{i,j+1}) \\ &\le -\frac{\varepsilon\Delta t}{3}(x_{i+1} - x_{i-1})v_\varepsilon'''(x_{i,j+1}), \end{aligned} \tag{54}$$

Now, since $x_{i+1} - x_{i-1} \le 2N^{-1}$ holds for all choice of piecewise uniform meshes, therefore we have

$$\begin{aligned} |L^N_{x,\varepsilon}(v^N_\varepsilon - v_\varepsilon)(x_{i,j+1})| &\le \left| -\frac{\varepsilon\Delta t}{3}(x_{i+1} - x_{i-1})\right| \, \| v_\varepsilon'''(x_{i,j+1}) \|_\infty \\ &\le \left|\frac{\varepsilon\Delta t}{3}CN^{-1}\right|(C(1 + \varepsilon^{-1/2}E(x,\theta))) \\ &\le C\ N^{-1}M^{-1} \end{aligned} \tag{55}$$

by an apllication of the preceeding lemma 7.2, we have

$$|(v^N_\varepsilon - v_\varepsilon)(x_{i,j+1})| \le |L^N_{x,\varepsilon}(v^N_\varepsilon - v_\varepsilon)(x_{i,j+1})| \le C\ N^{-1}M^{-1}, \tag{56}$$

hence the result follows.

C. Error in Singular Component

Lemma 7.4 For all $N \geq 8$, at each time step $(j+1)$ and mesh point $x_i \in \overline{\Omega}_x^N$ the singular component of the error satisfies the estimates

$$|(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| \leq CN^{-1}M^{-1}\ln N. \tag{57}$$

Proof. To estimate the error in the singular component of the solution the argument depends on whether

$$\delta = \frac{1}{4} \quad \text{or} \quad \delta = \frac{\sqrt{\varepsilon}}{\theta}\ln N$$

when the mesh is uniform and $\frac{\sqrt{\varepsilon}}{\theta} \ln N \geq 1/4$

The classical argument used in the estimation of the smooth component of the error leads to

$$\begin{aligned} L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1}) &= (L_{x,\varepsilon}^N w_\varepsilon^N - L_{x,\varepsilon}^N w_\varepsilon)(x_{i,j+1}) \\ &= (0 - L_{x,\varepsilon}^N w_\varepsilon)(x_{i,j+1}) \\ &= (L_{x,\varepsilon} - L_{x,\varepsilon}^N) w_\varepsilon(x_{i,j+1}) \\ &= -\varepsilon\Delta t\left(\frac{\partial^2}{\partial x^2} - D^+D^-\right) w_\varepsilon(x_{i,j+1}) \\ &\leq -\frac{\varepsilon\Delta t}{3}(x_{i+1} - x_{i-1}) w_\varepsilon'''(x_{i,j+1}). \end{aligned} \tag{58}$$

Taking modulus both the sides we have,

$$|L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| \leq |-\frac{\varepsilon\Delta t}{3}(x_{i+1} - x_{i-1})| \, \| w_\varepsilon'''(x_{i,j+1}) \|_\infty.$$

Since $(x_{i+1} - x_{i-1}) \leq 2N^{-1}$, and using the fact that third derivative w_ε''' is bounded by $C\varepsilon^{-3/2}E(x,\theta)$ we have,

$$\begin{aligned} |L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| &\leq |\frac{\varepsilon\Delta t}{3} 2N^{-1}| C\varepsilon^{-3/2}E(x,\theta) \\ &= \frac{2\Delta t}{3\sqrt{\varepsilon}} C\, N^{-1}E(x,\theta). \end{aligned} \tag{59}$$

But in this case $\frac{1}{\sqrt{\varepsilon}} \leq \frac{4}{\theta} \ln N$ and so by the preceding inequality, we obtained

$$|L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| \leq \frac{8\Delta t \ln N}{3\theta} C\, N^{-1}E(x,\theta) \leq C\, N^{-1}M^{-1}\ln N. \tag{60}$$

Now an application of lemma 7.2 to the function $(w_\varepsilon^N - w_\varepsilon)$, yields

$$\begin{aligned} |(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| &\leq |L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| \\ &\leq C\, N^{-1}M^{-1}\ln N. \end{aligned} \tag{61}$$

When the mesh is piecewise uniform with the mesh spacing $2(1-2\delta)/N$ *in the subintervals* $[\delta, 1-\delta]$ *and* $4\delta/N$ *in each of the subintervals* $[0,\delta]$ *and* $[1-\delta, 1]$. A somewhat different argument is used to bound $|(w_\varepsilon^N - w_\varepsilon)|$ depending on which mesh spacing occurs. For x_i lying in the open subintervals $(0,\delta)$ and $(1-\delta, 1)$. A similar classical argument used in the preceding case, leads to

$$|L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| \leq |\frac{\varepsilon\Delta t}{3}(x_{i+1} - x_{i-1})| \, \| w_\varepsilon'''(x_{i,j+1}) \|_\infty \leq \frac{4\Delta t\delta}{3\sqrt{\varepsilon}} C\, N^{-1}E(x,\theta).$$

since the mesh width is $4\delta/N$ and the fact that w_ε''' is bounded by $C\varepsilon^{-3/2}E(x,\theta)$. But since $\delta = \frac{\sqrt{\varepsilon}\ln N}{\theta}$, we have

$$|L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| \leq \frac{4\Delta t \ln N}{3\theta} C\, N^{-1}E(x,\theta) = CN^{-1}M^{-1}\ln N. \tag{62}$$

On the other hand, for x_i lying in the closed subinterval $[\delta, 1-\delta]$ the local truncation error of the singular part of the solution is estimated as follows

$$\begin{aligned} |L_{x,\varepsilon}^N(w_\varepsilon^N - w_\varepsilon)(x_{i,j+1})| &= \varepsilon\Delta t |(D^+D^- w_\varepsilon - w_\varepsilon'')(x_{i,j+1})| \\ &\leq 2\varepsilon\Delta t \max_{x \in [x_{i-1,j+1}, x_{i+1,j+1}]} |w_\varepsilon''(x)| \\ &\leq 2\Delta t \max_{x \in [x_{i-1,j+1}, x_{i+1,j+1}]} CE(x,\theta) \\ &= CM^{-1}\begin{cases} e^{-x_{i-1,j+1}\sqrt{\frac{\theta}{\varepsilon}}}, & \text{if} x_{i,j+1} \leq \frac{1}{2} \\ e^{-(1-x_{i,j+1})\sqrt{\frac{\theta}{\varepsilon}}}, & \text{if} x_{i,j+1} \geq \frac{1}{2} \end{cases} \end{aligned} \tag{63}$$

since $|(D^+D^- w_\varepsilon(x_{i,j+1})| \leq \max_{x \in [x_{i-1,j+1}, x_{i+1,j+1}]} |w_\varepsilon''(x)|$ and the fact that the second derivative of w_ε is bounded by $C\varepsilon^{-1}E(x,\theta)$.

In the case $x_{i,j+1} \leq 1/2$ then $x_{i,j+1} \geq \delta$. If the former is true then $x_{i-1,j+1} \geq \delta$ either $x_{i,j+1} > \delta$ or $x_{i,j+1} = \delta$ and so

$$e^{-x_{i-1,j+1}\sqrt{\frac{\theta}{\varepsilon}}} \leq e^{-\delta\sqrt{\frac{\theta}{\varepsilon}}} = e^{-\ln N} = N^{-1}. \tag{64}$$

If the later is true then $x_{i,j+1} = \delta$, and so $x_{i-1,j+1} = \delta - \frac{4\delta}{N}$. Thus

$$e^{-(1-x_{i,j+1})\sqrt{\frac{\theta}{\varepsilon}}} = e^{-\delta\sqrt{\frac{\theta}{\varepsilon}}} \cdot e^{\frac{4\delta}{N}\sqrt{\frac{\theta}{\varepsilon}}} = e^{-\ln N} \cdot e^{4N^{-1}\ln N} = N^{-1} \cdot (N^{1/N})^4 \le CN^{-1} \qquad (65)$$

since $N^{1/N} \le C$ for all $N \ge 1$. It follows that

$$|L^N_{x,\varepsilon}(w^N_\varepsilon - w_\varepsilon)(x_{i,j+1})| \le CN^{-1}M^{-1}\ln N.$$

Applying lemma (7.2) to the mesh function $(w^N_\varepsilon - w_\varepsilon)$ leads to

$$|(w^N_\varepsilon - w_\varepsilon)(x_{i,j+1})| \le |L^N_{x,\varepsilon}(w^N_\varepsilon - w_\varepsilon)(x_{i,j+1})| \le C\ N^{-1}M^{-1}\ln N. \qquad (66)$$

the required estimate of the error in the singular component of the solution. This concludes the proof of the lemma.

By now we have proved the main result of this article.

Theorem 7.5 The fitted mesh finite difference method with the standard upwind finite difference operator and the piecewise uniform fitted mesh $\overline{\Omega}^N_x$, condensing at the boundary points, is ε-uniform for the continuous problem (1), (2) provided that δ is chosen to satisfy condition (38). Moreover the solution u_ε of the continuous problem (1), (2) and the solution u^N_ε of the discrete problem satisfies the following ε-uniform error estimates

$$\sup_{0<\varepsilon\le 1} \| u^N_\varepsilon - u_\varepsilon \|_{\infty,\overline{\Omega}^N_x} \le C(N^{-1}M^{-1}\ln N + M^{-1})$$

where C is a constant independent of ε.

Proof. Follows immediately from lemma (7.3) and lemma (7.4) and Theorem (4.2).

VIII. CONCLUSION

In this paper, we initiated the numerical study for singularly perturbed time delayed partial differential equation of convection diffusion type. To look after the solution of such type of problem, one encounters with two difficulties; one is due to the existence of the singular perturbation parameter multiplied in the diffusion term and another one is due to the presence of time delay. We sort out both the problems via generating some special type of meshes in both the directions i.e space as well as time. An extensive amount of theoretical study is carried out to obtain the stability and error estimates for the numerical scheme so constructed. These estimates shows that the numerical scheme is unconditional stable and converges independently with respect to singular perturbation parameter.

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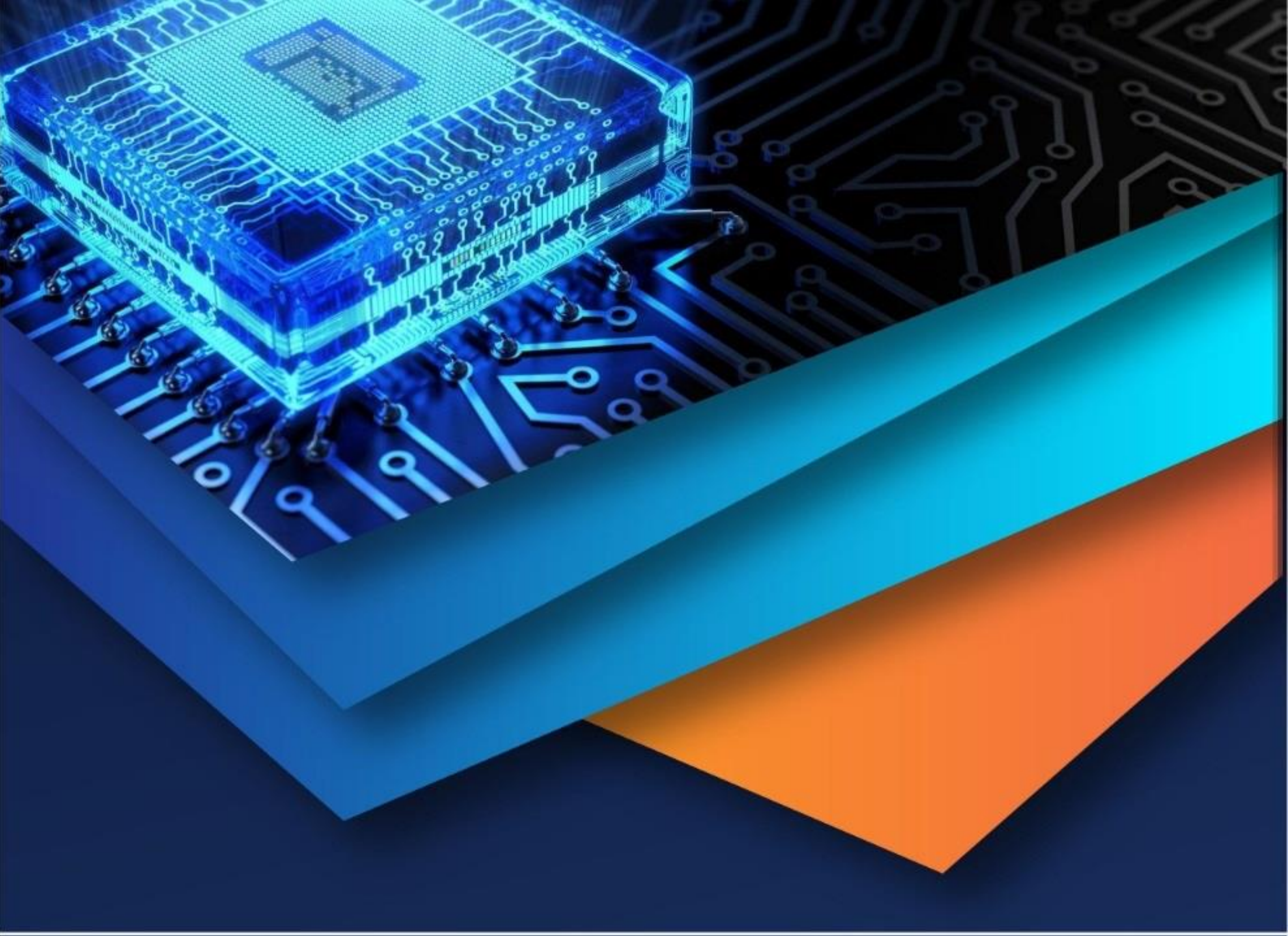

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