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Q-Hypergeometric Series and Their Transformation Formulae

Dr. Rajesh Pandey[1]
[1]Maharishi University of Information Technology Lucknow 22601, India.

Abstract: In this paper, making use of certain known summation formulae, an attempt has been made to establish transformation formulae, for q- hypergeometric series.
***Keywords*: Summation Formulae, Transformation Formulae, Hypergeometric Series, Identity, Inter-Series**

I. INTRODUCTION

In 1972 Verma [1] established the following *expansion* formula

$$\sum_{n=0}^{\infty}\frac{(-x)^n q^{n(n-1)/2}}{(q,\gamma q^n;q)_n}\sum_{k=0}^{\infty}\frac{(\alpha,\beta;q)_{n+k}}{(q,\gamma q^{2n+1};q)_k}B_{n+k}x^k\sum_{j=0}^{n}\frac{(q^{-n},\gamma q^n;q)}{(q,\alpha,\beta;q)_j}A_j(wq)^j = \sum_{n=0}^{\infty}A_n B_n\frac{(xw)^n}{(q;q)_n} \tag{1.1}$$

In this paper, making use of (1.1) and certain known summation formulae, an attempt has been made to establish transformation formulae for q-hypergeometric series.

II. NOTATIONS AND DEFINITIONS

The generalized basic hypergeometric function is defined as

$$_A\Phi_B\left[\begin{matrix}(a);q;z\\(b);q^i\end{matrix}\right] = \sum_{r=0}^{\infty}q^{\frac{ir(r-1)}{2}}\frac{\prod_{j=1}^{A}(a_j;q)_r z^r}{\prod_{j=1}^{B}(b_j;q)_r (q;q)_r} \tag{2.1}$$

Where

$$(a;q)_r = (1-a)(1-aq)\ldots(1-aq^{r-1});(a;q)_0 = 1, i>0, |q|<1, |z|<\infty \tag{2.2}$$

and for $i = 0$, max $(|q|,|z|) < 1$. Also stands for a sequences of A −parametrs of the form

$a_1, a_2, \ldots, a_A$ Type equation here.

We shall make use of following known summations

$$_4\Phi_3\left[\begin{matrix}a^2,a^2q,e^4q^{2n},q^{-2n};q^2;q^2\\a^4q^2,e^2,e^2q\end{matrix}\right] = \frac{(-q;q)_n(e^2/a^2;q)_n a^{2n}}{(e^2;q)_n(-a^2q;q)_n}. \tag{2.3}$$

$$_4\Phi_3\left[\begin{matrix}a^2,a^2q,e^4q^{2n},q^{-2n};q^2;q^2\\a^4,e^2q,e^2q^2\end{matrix}\right] = \frac{(-q;q)_n(e^2;q^2)_n(e^2q/a^2;q)_n a^{2n}}{(-a^2;q)_n(e^2;q)_n(e^2q^2;q^2)_n}, \tag{2.4}$$

III. MAIN RESULTS

We shall establish our main results

$$_{10}\Phi_9\left[\begin{matrix}-e^2,eiq,-eiq,eq,-eq,e^2/a^2,\alpha,-\alpha,\beta,-\beta;q;-\dfrac{e^4a^2q^2}{\alpha^2\beta^2}\\ei,-ei,-e,e,-a^2q,-e^2q/\alpha,e^2q/\alpha,-e^2q/\beta,e^2q/\beta;q^2\end{matrix}\right]$$

$$= \frac{(e^4q^2/\alpha^2\beta^2, e^4q^2; q^2)_\infty}{(e^4q^2/\alpha^2, e^4q^2/\beta^2; q^2)_\infty} \, {}_4\Phi_3 \left[\begin{matrix} a^2, a^2q, \alpha^2, \beta^2; q^2; \dfrac{e^4q^2}{\alpha^2\beta^2} \\ a^4q^2, e^2, e^2q \end{matrix} \right] \quad (3.1)$$

$${}_8\Phi_7 \left[\begin{matrix} -e^2, eiq, -eiq, e^2q/a^2, \alpha, -\alpha, \beta, -\beta; q; -\dfrac{e^4a^2q^2}{\alpha^2\beta^2} \\ ei, -ei, -a^2, -e^2q/\alpha, e^2q/\alpha, -e^2q/\beta, e^2q/\beta; q^2 \end{matrix} \right]$$

$$= \frac{(e^4q^2/\alpha^2\beta^2, e^4q^2; q^2)_\infty}{(e^4q^2/\alpha^2, e^4q^2/\beta^2; q^2)_\infty} \, {}_4\Phi_3 \left[\begin{matrix} a^2, a^2q, \alpha^2, \beta^2; q^2; \dfrac{e^4q^2}{\alpha^2\beta^2} \\ a^4, e^2q, e^2q^2 \end{matrix} \right] \quad (3.2$$

Proof of (3.1) and (3.2)

Replacing q, α, β by q^2, α^2, β^2 respectively and then choosing

$$A_j = \frac{(a^2, a^2q, \alpha^2, \beta^2; q^2)_j}{(a^4q^2, e^2, e^2q; q^2)_j}, \gamma = e^4, w = 1, B_n = 1,$$

$x = e^4q^2/\alpha^2\beta^2$ in (1.1) and making use of (2.3) and Gauss's summation formula in order to sum the inner-series in the left hand side we get (3.1) after some simplifications.

Similarly, replacing q, α, β by q^2, α^2, β^2 respectively and then choosing

$$A_j = \frac{(a^2, a^2q, \alpha^2, \beta^2; q^2)_j}{(a^4, e^2q, e^2q^2; q^2)_j}, w = 1, \gamma = e^4, B_n = 1, x = \frac{e^4q^2}{\alpha^2\beta^2}$$

In (1.1) and making use of use of (2.4) and Gauss's summation formula in order to sum the inner series in the left hand side we get (3.2) after some simplifications.

Taking $\alpha, \beta \to \infty$ in (3.1) we get

$$\sum_{r=0}^{\infty} \frac{(-e^2;q)_r (e^2/a^2;q)_r}{(q;q)_r (-a^2q;q)_r} \left(\frac{1-e^4q^{4r}}{1-e^4} \right) q^{3r(r-1)} (-e^4a^2q^2)^r$$

$$= (e^4q^2;q^2)_\infty \sum_{r=0}^{\infty} \frac{(a^2, a^2q; q^2)_r e^{4r} q^{2r^2}}{(q^2, a^4q^2, e^2, e^2q; q^2)_r} \quad (3.3)$$

Taking $a = 1$ and $e^4 = 1$ in (3.3) we obtain

$$\sum_{r=-\infty}^{\infty} (-)^r \, q^{r(3r-1)} = (q^2;q^2)_\infty, \quad (3.4)$$

Which on replacing q^2 by q gives the Euler's pentagonal identity:

$$\sum_{r=-\infty}^{\infty} (-)^r \, q^{r(3r-1)/2} = (q;q)_\infty,$$

Taking $a = 1$ and $e^4 = q^2$ in (3.3) we get another identity:

$$\sum_{r=0}^{\infty} (-)^r \, (1-q^{4r+2}) q^{r(3r+1)} = (q^2;q^2)_\infty. \quad (3.5)$$

Taking $a^2 = 1$ in (3.1) we obtain the following summation formula:

$$_5\Phi_4\begin{bmatrix}e^4, e^2q^2, -e^2q^2, \alpha^2, \beta^2; q^2; -e^4q^2/\alpha^2\beta^2 \\ e^2, -e^2, e^4q^2/\alpha^2, e^4q^2/\beta^2 ; q^2\end{bmatrix}$$

$$= \frac{(e^4q^2/\alpha^2\beta^2, e^4\,q^2; q^2)_\infty}{(e^4q^2/\alpha^2, e^4q^2/\beta^2; q^2)_\infty}. \qquad (3.6)$$

Taking $\alpha = e$ and $\beta = eq^{1/2}$ in (3.1) we get the following summation formula:

$$_4\Phi_3\begin{bmatrix}-e^2, eiq, -eiq, e^2/a^2; q; -a^2q \\ ei, -ei, -a^2q; q^2\end{bmatrix} = \frac{(-e^2q; q)_\infty}{(-a^2q; q)_\infty}. \qquad (3.7)$$

Taking $a \to 0$ in (3.7) we get:

$$\sum_{r=0}^{\infty}\frac{(-e^2; q)_r}{(q; q)_r}(1 + e^2q^{2r})e^{2r}q^{r(3r-1)/2} = (-e^2; q)_\infty \qquad (3.8)$$

Which for $e^2 = q$ yields:

$$\sum_{r=0}^{\infty}\frac{(-q; q)_r}{(q; q)_r}(1 + q^{2r+1})q^{r(3+1)/2} = (-q; q)_\infty \qquad (3.9)$$

Taking $\alpha, \beta \to \infty$ in (3.2) we get:

$$\sum_{r=0}^{\infty}\frac{(-e^2; q)_r(1 + e^2q^{2r})(e^2q/a^2; q)_r}{(q; q)_r(1 + e^2)(-a^2; q)_r}q^{3r(r-1)}(-e^4a^2q^2)^r$$

$$= (e^4q^2; q^2)_\infty\sum_{r=0}^{\infty}\frac{(a^2, a^2q; q^2)_r(e^4q^2)^r q^{2r(r-1)}}{(q^2, a^4, e^2q, e^2q^2; q^2)_r} \qquad (3.10)$$

For $a \to 1$, (3.10) gives:

$$\sum_{r=0}^{\infty}\frac{(-e^2, e^2q; q)_r(1 + e^2q^{2r})}{(q; q)_r(-1; q)_r(1 + e^2)}q^{3r(r-1)}(-e^4q^2)^r$$

$$= (e^2q^2; q^2)_\infty\left\{1 + \frac{1}{2}\sum_{r=1}^{\infty}\frac{(q; q^2)_r e^{4r}q^{2r^2}}{(q^2, e^2q, e^2q^2; q^2)_r}\right\} \qquad (3.11)$$

Taking $e^2 = 1$ in (3.11) we find :

$$\sum_{r=0}^{\infty}(1 + q^{2r})(-)^r q^{r(3r-1)} = (q^2; q^2)_\infty\left\{1 + \sum_{r=0}^{\infty}\frac{q^{2r^2}}{(q^2; q^2)_r^2}\right\},$$

Which by an appeal to Jacobi's triple product identity yields the well known identity (after replacing q^2 by q)

$$\sum_{r=0}^{\infty}\frac{q^{r^2}}{(q; q)_r^2} = \frac{1}{(q; q)_r} \qquad (3.12)$$

Similarly, several results can also be obtained.

IV. CONCLUSIONS

In this paper, transformation formulae for q-hypergeometric series have been established by using certain known summation formulae. Eight important results have been derived including Euler's pentagonal identity and Jacobi's triple product identity.

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