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Q-Hypergeometric Series and Their Transformation Formulae

Dr. Rajesh Pandey¹

¹Maharishi University of Information Technology Lucknow 22601, India.

Abstract: In this paper, making use of certain known summation formulae, an attempt has been made to establish transformation formulae, for q- hypergeometric series.

Keywords: Summation Formulae, Transformation Formulae, Hypergeometric Series, Identity, Inter-Series

I. INTRODUCTION

In 1972 Verma [1] established the following expansion formula

$$\sum_{n=0}^{\infty} \frac{(-x)^n q^{n(n-1)/2}}{(q, \gamma q^n; q)_n} \sum_{k=0}^{\infty} \frac{(\alpha, \beta; q)_{n+k}}{(q, \gamma q^{2n+1}; q)_k} B_{n+k} x^k \sum_{j=0}^{n} \frac{(q^{-n}, \gamma q^n; q)}{(q, \alpha, \beta; q)_j} A_j(wq)^j = \sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{(q; q)_n}$$
(1.1)

In this paper, making use of (1.1) and certain known summation formulae, an attempt has been made to establish transformation formulae for q-hypergeometric series.

II. NOTATIONS AND DEFINITIONS

The generalized basic hypergeometric function is defined as

$${}_{A}\Phi_{B} \boxtimes \begin{bmatrix} (a); q; z\\ (b); q^{i} \end{bmatrix} = \sum_{r=0}^{\infty} q^{\frac{ir(r-1)}{2}} \frac{\prod_{j=1}^{A} (a_{j}; q)_{r} z^{r}}{\prod_{j=1}^{B} (b_{j}; q)_{r} (q; q)_{r}}$$
(2.1)

Where

$$(a;q)_{r} = (1-a)(1-aq) \dots (1-aq^{r-1}); (a;q)_{0} = 1, i > 0, |q| < 1, |z| < \infty$$
(2.2)

and for i = 0, max (|q|, |z|) < 1. Also stands for a sequences of A – parameters of the form

$$a_1, a_2, \dots, a_A$$
 Type equation here.

We shall make use of following known summations

$${}_{4}\Phi_{3}\begin{bmatrix}a^{2},a^{2}q,e^{4}q^{2n},q^{-2n};q^{2};q^{2}\\a^{4}q^{2},e^{2},e^{2}q\end{bmatrix} = \frac{(-q;q)_{n}(e^{2}/a^{2};q)_{n}a^{2n}}{(e^{2};q)_{n}(-a^{2}q;q)_{n}}.$$
(2.3)

$${}_{4}\Phi_{3}\begin{bmatrix}a^{2},a^{2}q,e^{4}q^{2n},q^{-2n};q^{2};q^{2}\\a^{4},e^{2}q,e^{2}q^{2}\end{bmatrix} = \frac{(-q;q)_{n}(e^{2};q^{2})_{n}(e^{2}q/a^{2};q)_{n}a^{2n}}{(-a^{2};q)_{n}(e^{2};q)_{n}(e^{2}q^{2};q^{2})_{n}},$$
(2.4)

III. MAIN RESULTS

We shall establish our main results

$${}_{10}\Phi_{9}\begin{bmatrix}-e^{2}, eiq, -eiq, eq, -eq, e^{2}/a^{2}, \alpha, -\alpha, \beta, -\beta; q; -\frac{e^{4}a^{2}q^{2}}{\alpha^{2}\beta^{2}}\\ei, -ei, -e, e, -a^{2}q, -e^{2}q/\alpha, e^{2}q/\alpha, -e^{2}q/\beta, e^{2}q/\beta; q^{2}\end{bmatrix}$$



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$$= \frac{(e^{4}q^{2}/\alpha^{2}\beta^{2}, e^{4}q^{2}; q^{2})_{\infty}}{(e^{4}q^{2}/\alpha^{2}, e^{4}q^{2}/\beta^{2}; q^{2})_{\infty}} {}_{4}\Phi_{3} \begin{bmatrix} a^{2}, a^{2}q, \alpha^{2}, \beta^{2}; q^{2}; \frac{e^{4}q^{2}}{\alpha^{2}\beta^{2}} \\ a^{4}q^{2}, e^{2}, e^{2}q \end{bmatrix}$$

$$= \frac{(e^{4}q^{2}/\alpha^{2}\beta^{2}, e^{4}q^{2}; q^{2})_{\infty}}{(e^{4}q^{2}/\alpha^{2}, e^{2}q/\alpha, e^{2}q/\alpha, -e^{2}q/\beta, e^{2}q/\beta; q^{2}]}$$

$$= \frac{(e^{4}q^{2}/\alpha^{2}\beta^{2}, e^{4}q^{2}; q^{2})_{\infty}}{(e^{4}q^{2}/\alpha^{2}, e^{4}q^{2}/\beta^{2}; q^{2})_{\infty}} {}_{4}\Phi_{3} \begin{bmatrix} a^{2}, a^{2}q, \alpha^{2}, \beta^{2}; q^{2}; \frac{e^{4}q^{2}}{\alpha^{2}\beta^{2}} \\ a^{4}, e^{2}q, e^{2}q^{2} \end{bmatrix}$$

$$(3.1)$$

Proof of (3.1) and (3.2)

Replacing $q_{,\alpha,\beta}$ by $q^{2}_{,\alpha^{2},\beta^{2}}$ respectively and then choosing

$$A_{j} = \frac{(a^{2}, a^{2}q, \alpha^{2}, \beta^{2}; q^{2})_{j}}{(a^{4}q^{2}, e^{2}, e^{2}q; q^{2})_{j}}, \gamma = e^{4}, w = 1, B_{n} = 1,$$

 $x = e^4 q^2 / \alpha^2 \beta^2$ in (1.1) and making use of (2.3) and Gauss's summation formula in order to sum the inner-series in the left hand side we get (3.1) after some simplifications.

Similarly, replacing $q_1 \alpha_1 \beta$ by $q^2_1 \alpha^2_1 \beta^2$ respectively and then choosing

$$A_{j} = \frac{(a^{2}, a^{2}q, \alpha^{2}, \beta^{2}; q^{2})_{j}}{(a^{4}, e^{2}q, e^{2}q^{2}; q^{2})_{j}}, w = 1, \gamma = e^{4}, B_{n} = 1, x = \frac{e^{4}q^{2}}{\alpha^{2}\beta^{2}}$$

In (1.1) and making use of use of (2.4) and Gauss's summation formula in order to sum the inner series in the left hand side we get (3.2) after some simplifications.

Taking $\alpha, \beta \rightarrow \infty$ in (3.1) we get

$$\sum_{r=0}^{\infty} \frac{(-e^2;q)_r (e^2/a^2;q)_r}{(q;q)_r (-a^2q;q)_r} \left(\frac{1-e^4q^{4r}}{1-e^4}\right) q^{3r(r-1)} (-e^4a^2q^2)^r$$

$$= (e^4q^2;q^2)_{\infty} \sum_{r=0}^{\infty} \frac{(a^2,a^2q;q^2)_r e^{4r}q^{2r^2}}{(q^2,a^4q^2,e^2,e^2q;q^2)_r}$$
(3.3)

Taking a = 1 and $e^4 = 1$ in (3.3) we obtain

$$\sum_{r=-\infty}^{\infty} (-)^r q^{r(3r-1)} = (q^2; q^2)_{\infty}$$
(3.4)

Which on replacing q^2 by q gives the Euler's pentagonal identity:

$$\sum_{r=-\infty}^{\infty} (-)^r \, q^{r(3r-1)/2} = (q;q)_{\infty},$$

Taking a = 1 and $e^4 = q^2$ in (3.3) we get another identity:

$$\sum_{r=0}^{\infty} (-)^r (1 - q^{4r+2}) q^{r(3r+1)} = (q^2; q^2)_{\infty}.$$
(3.5)

Taking $a^2 = 1$ in (3.1) we obtain the following summation formula:



$${}_{5}\Phi_{4}\begin{bmatrix} e^{4}, e^{2}q^{2}, -e^{2}q^{2}, \alpha^{2}, \beta^{2}; q^{2}; -e^{4}q^{2}/\alpha^{2}\beta^{2} \\ e^{2}, -e^{2}, e^{4}q^{2}/\alpha^{2}, e^{4}q^{2}/\beta^{2}; q^{2} \end{bmatrix}$$

$$=\frac{(e^{4}q^{2}/\alpha^{2}\beta^{2}, e^{4}q^{2}; q^{2})_{\infty}}{(e^{4}q^{2}/\alpha^{2}, e^{4}q^{2}/\beta^{2}; q^{2})_{\infty}}.$$
(3.6)

Taking a = e and $\beta = eq^{1/2}$ in (3.1) we get the following summation formula:

$${}_{4}\Phi_{3}\begin{bmatrix} -e^{2}, eiq, -eiq, e^{2}/a^{2}; q; -a^{2}q \\ ei, -ei, -a^{2}q; q^{2} \end{bmatrix} = \frac{(-e^{2}q; q)_{\infty}}{(-a^{2}q; q)_{\infty}}.$$
(3.7)

Taking $a \rightarrow 0$ in (3.7) we get:

$$\sum_{r=0}^{\infty} \frac{(-e^2;q)_r}{(q;q)_r} (1 + e^2 q^{2r}) e^{2r} q^{r(3r-1)/2} = (-e^2;q)_{\infty}$$
(3.8)

Which for $e^2 = q$ yields:

$$\sum_{r=0}^{\infty} \frac{(-q;q)_r}{(q;q)_r} (1+q^{2r+1})q^{r(3+1)/2} = (-q;q)_{\infty}$$
(3.9)

Taking $\alpha, \beta \to \infty$ in (3.2) we get:

$$\sum_{r=0}^{\infty} \frac{(-e^2;q)_r (1+e^2q^{2r})(e^2q/a^2;q)_r}{(q;q)_r (1+e^2)(-a^2;q)_r} q^{3r(r-1)} (-e^4a^2q^2)^r$$

= $(e^4q^2;q^2)_{\infty} \sum_{r=0}^{\infty} \frac{(a^2,a^2q;q^2)_r (e^4q^2)^r q^{2r(r-1)}}{(q^2,a^4,e^2q,e^2q^2;q^2)_r}$ (3.10)

For $a \rightarrow 1$, (3.10) gives:

$$\sum_{r=0}^{\infty} \frac{(-e^2, e^2q; q)_r (1 + e^2q^{2r})}{(q; q)_r (-1; q)_r (1 + e^2)} q^{3r(r-1)} (-e^4q^2)^r$$

$$= (e^2q^2; q^2)_{\infty} \left\{ 1 + \frac{1}{2} \sum_{r=1}^{\infty} \frac{(q; q^2)_r e^{4r} q^{2r^2}}{(q^2, e^2q; q^2; q^2)_r} \right\}$$
(3.11)

Taking $e^2 = 1$ in (3.11) we find :

$$\sum_{r=0}^{\infty} (1+q^{2r})(-)^r q^{r(3r-1)} = (q^2;q^2)_{\infty} \left\{ 1 + \sum_{r=0}^{\infty} \frac{q^{2r^2}}{(q^2;q^2)_r^2} \right\},$$

Which by an appeal to Jacobi's triple product identity yields the well known identity (after replacing q^2 by q)

$$\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q;q)_r^2} = \frac{1}{(q;q)_r}$$
(3.12)

Similarly, several results can also be obtained.

IV. CONCLUSIONS

In this paper, transformation formulae for q-hypergeometric series have been established by using certain known summation formulae. Eight important results have been derived including Euler's pentagonal identity and Jacobi's triple product identity.

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