# Q-Hypergeometric Series and Their Transformation Formulae 

Dr. Rajesh Pandey ${ }^{1}$<br>${ }^{1}$ Maharishi University of Information Technology Lucknow 22601, India.


#### Abstract

In this paper, making use of certain known summation formulae, an attempt has been made to establish transformation formulae, for $q$ - hypergeometric series. Keywords: Summation Formulae, Transformation Formulae, Hypergeometric Series, Identity, Inter-Series


## I. INTRODUCTION

In 1972 Verma [1] established the following expansion formula
$\sum_{n=0}^{\infty} \frac{(-x)^{n} q^{n(n-1) / 2}}{\left(q, \gamma q^{n} ; q\right)_{n}} \sum_{k=0}^{\infty} \frac{(\alpha, \beta ; q)_{n+k}}{\left(q, \gamma q^{2 n+1} ; q\right)_{k}} B_{n+k} x^{k} \sum_{j=0}^{n} \frac{\left(q^{-n}, \gamma q^{n} ; q\right)}{(q, \alpha, \beta ; q)_{j}} A_{j}(w q)^{j}=\sum_{n=0}^{\infty} A_{n} B_{n} \frac{(x w)^{n}}{(q ; q)_{n}}$
In this paper, making use of (1.1) and certain known summation formulae, an attempt has been made to establish transformation formulae for q-hypergeometric series.

## II. NOTATIONS AND DEFINITIONS

The generalized basic hypergeometric function is defined as

$$
{ }_{A} \Phi_{B \text { 目 }}\left[\begin{array}{c}
(a) ; q ; z  \tag{2.1}\\
(b) ; q^{i}
\end{array}\right]=\sum_{r=0}^{\infty} q^{\frac{i r(r-1)}{2}} \frac{\prod_{j=1}^{A}\left(a_{j} ; q\right)_{r} z^{r}}{\prod_{j=1}^{B}\left(b_{j} ; q\right)_{r}(q ; q)_{r}}
$$

Where

$$
\begin{equation*}
(a ; q)_{r}=(1-a)(1-a q) \ldots\left(1-a q^{r-1}\right) ;(a ; q)_{0}=1, i>0,|q|<1,|z|<\infty \tag{2.2}
\end{equation*}
$$

and for $i=0, \max (|q|,|z|)<1$. Also stands for a sequences of A-parametrs of the form

$$
a_{1}, a_{2}, \ldots, a_{A} \text { Type equation here. }
$$

We shall make use of following known summations

$$
\begin{align*}
& { }_{4} \Phi_{3}\left[\begin{array}{l}
a^{2}, a^{2} q, e^{4} q^{2 n}, q^{-2 n} ; q^{2} ; q^{2} \\
a^{4} q^{2}, e^{2}, e^{2} q
\end{array}\right]=\frac{(-q ; q)_{n}\left(e^{2} / a^{2} ; q\right)_{n} a^{2 n}}{\left(e^{2} ; q\right)_{n}\left(-a^{2} q ; q\right)_{n}}  \tag{2.3}\\
& { }_{4} \Phi_{3}\left[\begin{array}{l}
a^{2}, a^{2} q, e^{4} q^{2 n}, q^{-2 n} ; q^{2} ; q^{2} \\
a^{4}, e^{2} q, e^{2} q^{2}
\end{array}\right]=\frac{(-q ; q)_{n}\left(e^{2} ; q^{2}\right)_{n}\left(e^{2} q / a^{2} ; q\right)_{n} a^{2 n}}{\left(-a^{2} ; q\right)_{n}\left(e^{2} ; q\right)_{n}\left(e^{2} q^{2} ; q^{2}\right)_{n}} \tag{2.4}
\end{align*}
$$

## III. MAIN RESULTS

We shall establish our main results

$$
{ }_{10} \Phi_{9}\left[\begin{array}{l}
-e^{2}, e i q,-e i q, e q,-e q, e^{2} / a^{2}, \alpha,-\alpha, \beta,-\beta ; q ;-\frac{e^{4} a^{2} q^{2}}{\alpha^{2} \beta^{2}} \\
e i,-e i,-e, e,-a^{2} q,-e^{2} q / \alpha, e^{2} q / \alpha,-e^{2} q / \beta, e^{2} q / \beta ; q^{2}
\end{array}\right]
$$

$$
\begin{align*}
& =\frac{\left(e^{4} q^{2} / \alpha^{2} \beta^{2}, e^{4} q^{2} ; q^{2}\right)_{\infty}}{\left(e^{4} q^{2} / \alpha^{2}, e^{4} q^{2} / \beta^{2} ; q^{2}\right)_{\infty}}{ }_{4} \Phi_{3}\left[\begin{array}{l}
a^{2}, a^{2} q, \alpha^{2}, \beta^{2} ; q^{2} ; \frac{e^{4} q^{2}}{\alpha^{2} \beta^{2}} \\
a^{4} q^{2}, e^{2}, e^{2} q
\end{array}\right]  \tag{3.1}\\
& { }_{8} \Phi_{7}\left[\begin{array}{l}
-e^{2}, e i q,-e i q, e^{2} q / a^{2}, \alpha,-\alpha, \beta,-\beta ; q ;-\frac{e^{4} a^{2} q^{2}}{\alpha^{2} \beta^{2}} \\
e i,-e i,-a^{2},-e^{2} q / \alpha, e^{2} q / \alpha,-e^{2} q / \beta, e^{2} q / \beta ; q^{2}
\end{array}\right] \\
& =\frac{\left(e^{4} q^{2} / \alpha^{2} \beta^{2}, e^{4} q^{2} ; q^{2}\right)_{\infty}}{\left(e^{4} q^{2} / \alpha^{2}, e^{4} q^{2} / \beta^{2} ; q^{2}\right)_{\infty}} \Phi_{3}\left[\begin{array}{l}
a^{2}, a^{2} q, \alpha^{2}, \beta^{2} ; q^{2} ; \frac{e^{4} q^{2}}{\alpha^{2} \beta^{2}} \\
a^{4}, e^{2} q, e^{2} q^{2}
\end{array}\right]
\end{align*}
$$

Proof of (3.1) and (3.2)
Replacing $q, \alpha, \beta$ by $q^{2}, \alpha^{2}, \beta^{2}$ respectively and then choosing
$A_{j}=\frac{\left(a^{2}, a^{2} q, \alpha^{2}, \beta^{2} ; q^{2}\right)_{j}}{\left(a^{4} q^{2}, e^{2}, e^{2} q ; q^{2}\right)_{j}}, \gamma=e^{4}, w=1, B_{n}=1$,
$x=e^{4} q^{2} / \alpha^{2} \beta^{2}$ in (1.1) and making use of (2.3) and Gauss's summation formula in order to sum the inner-series in the left hand side we get (3.1) after some simplifications.

Similarly, replacing $q, \alpha, \beta$ by $q^{2}, \alpha^{2}, \beta^{2}$ respectively and then choosing
$A_{j}=\frac{\left(a^{2}, a^{2} q, \alpha^{2}, \beta^{2} ; q^{2}\right)_{j}}{\left(a^{4}, e^{2} q, e^{2} q^{2} ; q^{2}\right)_{j}}, w=1, \gamma=e^{4}, B_{n}=1, x=\frac{e^{4} q^{2}}{\alpha^{2} \beta^{2}}$
In (1.1) and making use of use of (2.4) and Gauss's summation formula in order to sum the inner series in the left hand side we get (3.2) after some simplifications.

Taking $\alpha, \beta \rightarrow \infty$ in (3.1) we get
$\sum_{r=0}^{\infty} \frac{\left(-e^{2} ; q\right)_{r}\left(e^{2} / a^{2} ; q\right)_{r}}{(q ; q)_{r}\left(-a^{2} q ; q\right)_{r}}\left(\frac{1-e^{4} q^{4 r}}{1-e^{4}}\right) q^{3 r(r-1)}\left(-e^{4} a^{2} q^{2}\right)^{r}$
$=\left(e^{4} q^{2} ; q^{2}\right)_{\infty} \sum_{r=0}^{\infty} \frac{\left(a^{2}, a^{2} q ; q^{2}\right)_{r} e^{4 r} q^{2 r^{2}}}{\left(q^{2}, a^{4} q^{2}, e^{2}, e^{2} q ; q^{2}\right)_{r}}$
Taking $a=1$ and $e^{4}=1$ in (3.3) we obtain

$$
\begin{equation*}
\sum_{r=-\infty}^{\infty}(-)^{r} q^{r(3 r-1)}=\left(q^{2} ; q^{2}\right)_{\infty} \tag{3.4}
\end{equation*}
$$

Which on replacing $q^{2}$ by $q$ gives the Euler's pentagonal identity:

$$
\sum_{r=-\infty}^{\infty}(-)^{r} q^{r(3 r-1) / 2}=(q ; q)_{\infty}
$$

Taking $a=1$ and $e^{4}=q^{2}$ in (3.3) we get another identity:

$$
\begin{equation*}
\sum_{r=0}^{\infty}(-)^{r}\left(1-q^{4 r+2}\right) q^{r(3 r+1)}=\left(q^{2} ; q^{2}\right)_{\infty} \tag{3.5}
\end{equation*}
$$

Taking $a^{2}=1$ in (3.1) we obtain the following summation formula:

$$
\begin{align*}
& { }_{5} \Phi_{4}\left[\begin{array}{l}
e^{4}, e^{2} q^{2},-e^{2} q^{2}, \alpha^{2}, \beta^{2} ; q^{2} ;-e^{4} q^{2} / \alpha^{2} \beta^{2} \\
e^{2},-e^{2}, e^{4} q^{2} / \alpha^{2}, e^{4} q^{2} / \beta^{2} ; q^{2}
\end{array}\right] \\
& =\frac{\left(e^{4} q^{2} / \alpha^{2} \beta^{2}, e^{4} q^{2} ; q^{2}\right)_{\infty}}{\left(e^{4} q^{2} / \alpha^{2}, e^{4} q^{2} / \beta^{2} ; q^{2}\right)_{\infty}} . \tag{3.6}
\end{align*}
$$

Taking $a=e$ and $\beta=e q^{1 / 2}$ in (3.1) we get the following summation formula:

$$
{ }_{4} \Phi_{3}\left[\begin{array}{l}
-e^{2}, e i q,-e i q, e^{2} / a^{2} ; q ;-a^{2} q  \tag{3.7}\\
e i,-e i,-a^{2} q ; q^{2}
\end{array}\right]=\frac{\left(-e^{2} q ; q\right)_{\infty}}{\left(-a^{2} q ; q\right)_{\infty}} .
$$

Taking $a \rightarrow 0$ in (3.7) we get:
$\sum_{r=0}^{\infty} \frac{\left(-e^{2} ; q\right)_{r}}{(q ; q)_{r}}\left(1+e^{2} q^{2 r}\right) e^{2 r} q^{r(3 r-1) / 2}=\left(-e^{2} ; q\right)_{\infty}$
Which for $e^{2}=q$ yields:
$\sum_{r=0}^{\infty} \frac{(-q ; q)_{r}}{(q ; q)_{r}}\left(1+q^{2 r+1}\right) q^{r(3+1) / 2}=(-q ; q)_{\infty}$
Taking $\alpha, \beta \rightarrow \infty$ in (3.2) we get:
$\sum_{r=0}^{\infty} \frac{\left(-e^{2} ; q\right)_{r}\left(1+e^{2} q^{2 r}\right)\left(e^{2} q / a^{2} ; q\right)_{r}}{(q ; q)_{r}\left(1+e^{2}\right)\left(-a^{2} ; q\right)_{r}} q^{3 r(r-1)}\left(-e^{4} a^{2} q^{2}\right)^{r}$
$=\left(e^{4} q^{2} ; q^{2}\right)_{\infty} \sum_{r=0}^{\infty} \frac{\left(a^{2}, a^{2} q ; q^{2}\right)_{r}\left(e^{4} q^{2}\right)^{r} q^{2 r(r-1)}}{\left(q^{2}, a^{4}, e^{2} q, e^{2} q^{2} ; q^{2}\right)_{r}}$
For $a \rightarrow 1$, (3.10) gives:
$\sum_{r=0}^{\infty} \frac{\left(-e^{2}, e^{2} q ; q\right)_{r}\left(1+e^{2} q^{2 r}\right)}{(q ; q)_{r}(-1 ; q)_{r}\left(1+e^{2}\right)} q^{3 r(r-1)}\left(-e^{4} q^{2}\right)^{r}$
$=\left(e^{2} q^{2} ; q^{2}\right)_{\infty}\left\{1+\frac{1}{2} \sum_{r=1}^{\infty} \frac{\left(q ; q^{2}\right)_{r} e^{4 r} q^{2 r^{2}}}{\left(q^{2}, e^{2} q, e^{2} q^{2} ; q^{2}\right)_{r}}\right\}$
Taking $e^{2}=1$ in (3.11) we find:
$\sum_{r=0}^{\infty}\left(1+q^{2 r}\right)(-)^{r} q^{r(3 r-1)}=\left(q^{2} ; q^{2}\right)_{\infty}\left\{1+\sum_{r=0}^{\infty} \frac{q^{2 r^{2}}}{\left(q^{2} ; q^{2}\right)_{r}^{2}}\right\}$,
Which by an appeal to Jacobi's triple product identity yields the well known identity (after replacing $q^{2}$ by $q$ )
$\sum_{r=0}^{\infty} \frac{q^{r^{2}}}{(q ; q)_{r}^{2}}=\frac{1}{(q ; q)_{r}}$
Similarly, several results can also be obtained.

## IV. CONCLUSIONS

In this paper, transformation formulae for q-hypergeometric series have been established by using certain known summation formulae. Eight important results have been derived including Euler's pentagonal identity and Jacobi's triple product identity.

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